SECOND ORDER FAMILIES OF SPECIAL LAGRANGIAN 3-FOLDS

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Abstract. A second order family of special Lagrangian submanifolds of $\mathbb{C}^m$ is a family characterized by the satisfaction of a set of pointwise conditions on the second fundamental form. For example, the set of ruled special Lagrangian submanifolds of $\mathbb{C}^3$ is characterized by a single algebraic equation on the second fundamental form.

While the 'generic' set of such conditions turns out to be incompatible, i.e., there are no special Lagrangian submanifolds that satisfy them, there are many interesting sets of conditions for which the corresponding family is unexpectedly large. In some cases, these geometrically defined families can be described explicitly, leading to new examples of special Lagrangian submanifolds. In other cases, these conditions characterize already known families in a new way. For example, the examples of Lawlor-Harvey constructed for the solution of the angle conjecture and recently generalized by Joyce turn out to be a natural and easily described second order family.

Contents

1. Introduction 2
1.1. Special Lagrangian geometry 4
1.2. Special Kähler reduction 5
2. The Structure Equations 6
2.1. The special coframe bundle 6
2.2. The structure equations 7
2.3. Special Lagrangian submanifolds 7
3. Second Order Families 9
3.1. The second fundamental form as a cubic 9
3.2. Continuous symmetry 12
3.3. A_4 symmetry 15
3.4. S_3 symmetry 16
3.5. $\mathbb{Z}_2$ symmetry 21
3.6. $\mathbb{Z}_3$ symmetry 26
3.7. The ruled family 29
References 35

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1. Introduction

The study of special Lagrangian submanifolds was introduced by Harvey and Lawson in §III of their fundamental paper [12] on calibrated geometries. They analyzed the local and global geometry of these submanifolds in flat complex m-space, constructing many interesting examples and proving local existence theorems.

Several important classes of examples of these submanifolds have been constructed since then, mainly with an eye to applications in the theory of calibrations or minimizing submanifolds. Particularly important was the construction by Lawlor [19] of a special Lagrangian manifold asymptotic to a pair of planes that violate the angle criterion, thus proving that such a pair of planes is not area-minimizing. See also Harvey [11] for a thorough account of this example and its applications.

The deformation theory of compact examples in Calabi-Yau manifolds was studied in the 1990 thesis of R. McLean [21], who showed that the moduli space of compact special Lagrangian submanifolds of a given Calabi-Yau manifold is always a disjoint union of smooth manifolds.

Special Lagrangian geometry received renewed attention in 1996 when its role in mirror symmetry was discovered by Strominger, Yau, and Zaslow [23]. Since then, interest in special Lagrangian geometry has grown quite rapidly. The reader might consult [14] (for what is currently known about the moduli of compact special Lagrangian submanifolds), [7] (for some examples arising from algebraic geometry), [20] (for further information about mirror symmetry), [9] (for examples with large symmetry groups), [4] (for embedding a given real-analytic Riemannian 3-manifold as a special Lagrangian submanifold of a Calabi-Yau 3-fold), [15] (for some interesting speculations about how one might count the isolated special Lagrangian submanifolds), and [13] (for information about special Lagrangian cones in $\mathbb{C}^3$).

Still, the systematic exploration of special Lagrangian geometry seems to have hardly begun. The known explicit examples have largely been found by the well-known Ansatz of symmetry reduction or other special tricks.

The research that lead to this article was an attempt to classify families of special Lagrangian submanifolds that are characterized by invariant, differential geometric conditions, in particular, conditions on the second fundamental form of the special Lagrangian submanifold.

At least when the ambient space is flat, the lowest order invariant of a special Lagrangian submanifold is its second fundamental form. Now, for a Lagrangian submanifold of a linear symplectic vector space, the second fundamental form, usually defined as a quadratic form with values in the normal bundle, has a natural interpretation as a symmetric cubic form C on the submanifold, called the *fundamental cubic*. When the submanifold is special Lagrangian, it turns out that the trace of this cubic form with respect to the first fundamental form vanishes, but there are no further pointwise conditions on this cubic that are satisfied for all special Lagrangian submanifolds.

It is natural to ask whether one can obtain nontrivial families of special Lagrangian submanifolds by imposing pointwise conditions on the fundamental cubic. In the language of overdetermined systems of PDE, one would like to be able to

say whether there are any second order systems of PDE that are 'compatible' with the (first order) system that represents the special Lagrangian condition.[1]

The first task is to understand the space of pointwise invariants of a traceless cubic form under the special orthogonal group. For example, in dimension 3 (which is the case this article mainly considers), the space of traceless cubics is an irreducible SO(3)-module of dimension 7, so one would expect there to be four independent polynomial invariants.

However, the relations that one gets by imposing conditions on these invariants are generally singular at the cubics that have a nontrivial stabilizer under the action of SO(3). For comparison, consider the classical case of hypersurfaces in Euclidean space. The fundamental invariants are the principal curvatures, i.e., the eigenvalues of the second fundamental form with respect to the first fundamental form. These are smooth away from the (generalized) umbilic locus, i.e., the places where two or more of the principal curvatures come together. It is exactly at these places that the stabilizer of the second fundamental form in the orthogonal group is larger than the minimum possible stabilizer. Of course, the umbilic locus is also the place where moving frame adaptations generally run into trouble, unless one assumes that the multiplicities of the principal curvatures are constant.

There is a similar phenomenon in special Lagrangian geometry. In place of the umbilic locus, one looks that the places where the fundamental cubic has a nontrivial stabilizer,[2] and at the special Lagrangian submanifolds where the stabilizer of the cubic is nontrivial at the generic point. These are the special special Lagrangian submanifolds.

In this article, after making some general remarks to introduce the structure equations of special Lagrangian geometry, I classify the possible nontrivial SO(3)-stabilizers of traceless cubics in three variables. It turns out that the SO(3)-stabilizer of a nontrivial traceless cubic is isomorphic to either a copy of SO(2), the group A_4 of order 12, the group S_3 of order 6, the group $\mathbb{Z}_3$, the group $\mathbb{Z}_2$, or is trivial. I then consider, for each of the nontrivial subgroups G on this list, the problem of classifying the special Lagrangian 3-folds whose cubic form at each point has its stabilizer contain a copy of G.

For example, it turns out that the only special Lagrangian 3-folds in $\mathbb{C}^3$ whose cubic form has a continuous stabilizer at each point are the 3-planes and the SO(3)-invariant examples discovered by Harvey and Lawson.

There are no special Lagrangian 3-folds whose cubic stabilizer at at generic point is of type A_4, but the ones whose stabilizer at a generic point is of type S_3 turn out to be the austere special Lagrangian 3-folds and these are known to be the orthogonal products, the special Lagrangian cones, and the 'twisted' special Lagrangian cones.

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[1] Here, 'compatibility' is not strictly defined, but, roughly speaking, means that there exist at least as many (local) solutions to the overdetermined system as one would expect from a naïve 'equation counting' argument. A more precise description would involve concepts from exterior differential systems, such as involutivity, that will not be needed in this article.

[2] In contrast to the familiar case of hypersurfaces in Euclidean space, where the stabilizer, though generically finite, is always nontrivial, it turns out that the stabilizer at a generic point of the fundamental cubic of a 'generic' special Lagrangian 3-fold is trivial.

The special Lagrangian 3-folds with cubic stabilizer at a generic point isomorphic to $\mathbb{Z}_2$ turn out to be the examples discovered by Lawlor, extended by the work of Harvey and then Joyce.[3]

The special Lagrangian 3-folds whose cubic stabilizer at a generic point is isomorphic to $\mathbb{Z}_3$ turn out to be asymptotically conical and, indeed, turn out to be deformations, in a certain sense, of the special Lagrangian cones, as explained in the thesis of Haskins [13].

The above results are explained more fully in §3.

At the conclusion of §3, I consider a different type of invariant condition on the fundamental cubic, namely that, at every point $x \in L$, the degree 3 curve in $\mathbb{P}(T_xL)$ defined by the fundamental cubic have a real singular point. This is one semi-algebraic condition on the fundamental cubic. I show that the special Lagrangian 3-folds with this property are exactly the ruled special Lagrangian 3-folds.

Moreover, while the most general known family of ruled special Lagrangian 3-folds up until now was one discovered by Borisenko [1] and that depends on four functions of one variable (in the sense of exterior differential systems), I show that the full family depends on six functions of one variable. Moreover, I show that, when one interprets the ruled special Lagrangian 3-folds as surfaces in the space of lines in $\mathbb{C}^3$, the surfaces that one obtains are simply the ones that are holomorphic with respect to a canonical Levi-flat, almost CR-structure on the space of lines. This interpretation has several implications for the structure of ruled special Lagrangian 3-folds, among them being that any real-analytic ruled surface in $\mathbb{C}^3$ on which the Kähler form vanishes lies in a (essentially unique) ruled special Lagrangian 3-fold. Moreover, a special Lagrangian 3-fold is ruled if and only if it contains a ruled surface.

It has to be said that the results of this article are only the first step in understanding the compatibility of the special Lagrangian condition with higher order conditions. Now that the 'umbilic' cases are understood, the serious work on the 'generic' case can be undertaken. This will be reported on in a subsequent work.

Also, while, for the sake of brevity, this work has concerned itself (essentially exclusively) with the 3-dimensional case, there are obvious higher dimensional generalizations that need to be investigated and that should yield to the same or similar techniques.[4]

1.1. **Special Lagrangian geometry.**

In this article, a slightly more general notion of special Lagrangian geometry is adopted than is customary. The reader might compare this discussion with Harvey and Lawson's original article [12] or Harvey's more recent book [11].

1.1.1. *Special Kähler structures.* Let M be a complex m-manifold endowed with a Kähler form ω and a holomorphic volume form Υ. It is *not* assumed that Υ be parallel, or even of constant norm, with respect to the Levi-Civita connection associated to ω. The pair (ω, Υ) is said to define a *special Kähler structure* on M.

[3] I am indebted to Joyce for suggesting (by private communication) that the family that I had shown to exist in this case might be the Lawlor-Harvey-Joyce family. He was correct, and this saved me quite a bit of work in integrating the corresponding structure equations.

[4] My student, Marianty Ionel, has recently completed a study of the special Lagrangian 4-folds in $\mathbb{C}^4$ whose fundamental cubic has nontrivial symmetries.

1.1.2. *Special Lagrangian submanifolds.* A submanifold $L \subset M$ of real dimension m is said to be *Lagrangian*[5] if the pullback of ω to L vanishes.

Harvey and Lawson show [12, §III, Theorem 1.7] that for any Lagrangian submanifold $L \subset M$, the pullback of Υ to L can never vanish. A Lagrangian submanifold L is said to be *special Lagrangian* if the pullback of $\mathrm{Im}(\Upsilon)$ to L vanishes. When L is special Lagrangian, it has a canonical orientation for which $\mathrm{Re}(\Upsilon)$ pulls back to L to be a positive volume form, and this is the orientation that will be assumed throughout this article.

More generally, if λ is a complex number of unit modulus, one says that an oriented Lagrangian submanifold $L \subset M$ has *constant phase* λ if $\bar{\lambda}\,\Upsilon$ pulls back to L to be a (real-valued) positive volume form. Obviously, for any fixed λ, this notion is not significantly more general than the notion of special Lagrangian, so I will usually consider only special Lagrangian submanifolds in this article.

1.1.3. *The Calabi-Yau case.* When Υ is parallel with respect to the Levi-Civita connection associated to ω, the Kähler metric has vanishing Ricci tensor and the pair (ω, Υ) is said to define a *Calabi-Yau* structure on M. In this case, Harvey and Lawson show that any special Lagrangian submanifold $L \subset M$ is minimal. Moreover, if L is compact, it is absolutely minimizing in its homology class since it is then calibrated by $\mathrm{Re}(\Upsilon)$.

1.1.4. *Local existence.* Assume that ω is real-analytic with respect to the standard real-analytic structure on M that underlies its complex analytic structure. Harvey and Lawson show[12, §III, Theorem 5.5] that any real-analytic submanifold $N \subset M$ of dimension $m-1$ on which ω pulls back to be zero lies in a unique special Lagrangian submanifold $L \subset M$. (Although their result is stated only for the case of the standard flat special Kähler structure on $\mathbb{C}^m$, their proof is valid in the general case, provided one makes the necessary trivial notational changes.)

Thus, there are many special Lagrangian submanifolds locally, at least in the real-analytic category. By adapting arguments from [12, §III.2], one can also prove local existence of special Lagrangian submanifolds even without the assumption of real-analyticity. Instead, one uses local existence for an elliptic second order scalar equation.

1.1.5. *Deformations.* R. McLean [21] proved that, in the Calabi-Yau case, a compact special Lagrangian submanifold $L \subset M$ is a point in a smooth, finite dimensional moduli space $\mathcal{L}$ consisting of the special Lagrangian deformations of L and that the tangent space to $\mathcal{L}$ at L is isomorphic to the space of harmonic 1-forms on L.

McLean's argument makes no essential use of the assumption that Υ be ω-parallel. Instead, it is sufficient for the conclusion of McLean's theorem that Υ be closed (in fact, one only really needs that the imaginary part of Υ be closed.) For a related result, see [22].

1.2. **Special Kähler reduction.** One reason for considering the slightly wider notion of special Lagrangian geometry adopted here is that it is stable under the process of *reduction*, as explained in [8], [17], and [10].

[5] or ω-*Lagrangian* if there is any danger of confusion

Let (ω, Υ) be a special Kähler structure on M. A vector field X on M will be said to be an *infinitesimal symmetry* of the structure if the (locally defined) flow of X preserves both ω and Υ.

Suppose that X is an infinitesimal symmetry of (ω, Υ) and that X is, moreover, ω-Hamiltonian, i.e., that there exists a function H on M satisfying $X \lrcorner\, \omega = -\mathrm{d}H$. The flow lines of X are tangent to the level sets of H.

Say that a value $h \in \mathbb{R}$ is a *good* value for H if it is a regular value of H and if the flow of X on the level set $H^{-1}(h) \subset M$ is simple, i.e., there is a smooth manifold structure on the set M_h of flow lines of X in the level set $H^{-1}(h)$ so that the natural projection $\pi_h : H^{-1}(h) \to M_h$ is a smooth submersion. The (real) dimension of M_h is necessarily $2m{-}2$.

When h is good, there exists a unique 2-form ω_h on M_h for which $\pi_h^*(\omega_h)$ is the pullback of ω to $H^{-1}(h)$ and there exists a unique complex-valued $(m{-}1)$ form Υ_h on M_h for which $\pi_h^*(\Upsilon_h)$ is the pullback to $H^{-1}(h)$ of $X \lrcorner\, \Upsilon$.

It is trivial to verify that (ω_h, Υ_h) defines a special Kähler structure on M_h. Note, however, that, even if (ω, Υ) is Calabi-Yau, its reductions will generally *not* be Calabi-Yau. In fact, this happens only when the length of X is constant along the level set $H^{-1}(h)$.

If $L \subset H^{-1}(h)$ is a special Lagrangian submanifold that is tangent to the flow of X, then $L = \pi_h^{-1}(L_h)$ where $L_h \subset M_h$ is also special Lagrangian. Conversely, if $L_h \subset M_h$ is special Lagrangian, then $L = \pi_h^{-1}(L_h)$ is special Lagrangian in M.

This method of special Kähler reduction allows one to construct many examples of special Lagrangian submanifolds by starting with a Hamiltonian $(m{-}1)$-torus action and doing a series of reductions, leading to a 1-dimensional special Kähler manifold, where the integration problem is reduced to integrating a holomorphic 1-form on a Riemann surface.

2. The Structure Equations

The structure equations of a Kähler manifold adapted for special Lagrangian geometry can be found in [5], and will only be reviewed briefly here.

2.1. The special coframe bundle. The *standard special Kähler structure* on $\mathbb{C}^m$ is the one defined by

$$\omega_0 = \tfrac{\mathrm{i}}{2}\bigl(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\overline{z_m}\bigr) \qquad \text{and} \qquad \Upsilon_0 = \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_m\,, \tag{2.1}$$

where $z_1, \ldots, z_m$ are the usual complex linear coordinates on $\mathbb{C}^m$. The corresponding Kähler metric is, of course

$$g_0 = \mathrm{d}z_1 \circ \mathrm{d}\overline{z_1} + \cdots + \mathrm{d}z_m \circ \mathrm{d}\overline{z_m}\,. \tag{2.2}$$

Note that $\mathbb{R}^m \subset \mathbb{C}^m$ is a special Lagrangian subspace.

Let (ω, Υ) be a special Kähler structure on the complex m-manifold M. There is a unique positive function B on M that satisfies

$$\Upsilon \wedge \overline{\Upsilon} = \frac{2^m(-\mathrm{i})^{m^2}}{m!}\, B^2\, \omega^m\,. \tag{2.3}$$

A linear isomorphism $u : T_xM \to \mathbb{C}^m$ will be said to be a *special Kähler coframe* at x if it satisfies $\omega_x = u^*(\omega_0)$ and $\Upsilon_x = B(x)\, u^*(\Upsilon_0)$. Such a coframe is necessarily complex linear, i.e., satisfies $u(J_xv) = \mathrm{i}\, u(v)$ for all $v \in T_xM$, where $J_x : T_xM \to T_xM$ is the complex structure map.

The set of special Kähler coframes at x will be denoted P_x and is the fiber of a principal right $\mathrm{SU}(m)$-bundle $\pi : P \to M$, with right action given by $R_a(u) = a^{-1}\circ u$ for $a \in \mathrm{SU}(m)$.

As usual, the $\mathbb{C}^m$-valued, tautological 1-form ζ on P is defined by requiring that $\zeta_u = u\circ\pi'(u) : T_uP \to \mathbb{C}^m$ for $u \in P$. It satisfies $R_a^*(\zeta) = a^{-1}\,\zeta$ for $a \in \mathrm{SU}(m)$.

The components of ζ will be written as ζ_i for $1 \le i \le m$. The equations

$$\omega = \tfrac{\mathrm{i}}{2}\big(\zeta_1\wedge\overline{\zeta_1} + \cdots + \zeta_m\wedge\overline{\zeta_m}\big) \qquad \text{and} \qquad \Upsilon = B\,\zeta_1\wedge\ldots\wedge\zeta_m \tag{2.4}$$

hold on P, where, as is customary, I have omitted the π^*, thus implicitly embedding the differential forms on M into the differential forms on P via pullback.

Finally, there are functions $\mathbf{e} = (\mathbf{e}_i)$, where $\mathbf{e}_i : P \to TM$ is a bundle mapping satisfying $\zeta_i(\mathbf{e}_j) = \delta_{ij}$. In other words $\pi'(v) = \zeta_i(v)\,\mathbf{e}_i(u)$ for all $v \in T_uP$.

Remark 1 (The flat case). When $M = \mathbb{C}^m$ and $(\omega,\Upsilon) = (\omega_0,\Upsilon_0)$, it is customary to use the vector space (parallel) trivialization of the tangent bundle of $\mathbb{C}^m$ to identify all of the tangent spaces to the vector space $\mathbb{C}^m$ itself. In this case, the functions $\mathbf{e}_i$ will be regarded as vector-valued functions on $P \simeq \mathbb{C}^m \times \mathrm{SU}(m)$ and the basepoint projection will be denoted as $\mathbf{x} : P \to \mathbb{C}^m$. Then the above relations take on the more familiar 'moving frame' form

$$\mathrm{d}\mathbf{x} = \mathbf{e}_i\,\zeta_i\,,$$

and so on. The reader should have no trouble figuring out what is meant in context.

2.2. **The structure equations.** The Levi-Civita connection associated to the underlying Kähler structure on M is represented on P by a $\mathfrak{u}(m)$-valued 1-form $\psi = -\psi^* = (\psi_{i\bar{j}})$ that satisfies the *first structure equation*

$$\mathrm{d}\zeta_i = -\psi_{i\bar{j}}\wedge\zeta_j\,. \tag{2.5}$$

The equation $\mathrm{d}\Upsilon = 0$ implies

$$(\bar\partial - \partial)\big(\log B\big) = -\mathrm{i}\,\mathrm{d}^c(\log B) = \mathrm{tr}(\psi) = \psi_{ii}\,. \tag{2.6}$$

Note that (ω,Υ) is Calabi-Yau if and only if B is constant, i.e., if and only if ψ takes values in $\mathfrak{su}(m)$.

In the Calabi-Yau case, where Υ is parallel with respect to the Levi-Civita connection of ω, the relation $\psi_{ii} = 0$ holds. Moreover, the Calabi-Yau structure is locally equivalent to the standard structure if and only if the Levi-Civita connection of ω vanishes, i.e., if and only if $\mathrm{d}\psi = -\psi\wedge\psi$, which is known as the *second structure equation* of a flat Calabi-Yau space.

2.3. **Special Lagrangian submanifolds.** For the study of special Lagrangian submanifolds, it is convenient to separate the structure equations into real and imaginary parts. Thus, set $\zeta_i = \omega_i + \mathrm{i}\,\eta_i$ and $\psi_{i\bar{j}} = \alpha_{ij} + \mathrm{i}\,\beta_{ij}$. The first structure equations can then be written in the form

$$\begin{aligned} \mathrm{d}\,\omega_i &= -\alpha_{ij}\wedge\omega_j + \beta_{ij}\wedge\eta_j\,,\\ \mathrm{d}\,\eta_i &= -\beta_{ij}\wedge\omega_j - \alpha_{ij}\wedge\eta_j\,. \end{aligned} \tag{2.7}$$

where $\alpha_{ij} = -\alpha_{ji}$ and $\beta_{ij} = \beta_{ji}$. Note that (2.6) becomes $\beta_{ii} = -\mathrm{d}^c\big(\log B\big)$.

Let $L \subset M$ be a special Lagrangian submanifold. For $x \in L$, a special Kähler coframe $u : T_xM \to \mathbb{C}^m$ is said to be L-*adapted* if $u(T_xL) = \mathbb{R}^m \subset \mathbb{C}^m$ and $u : T_xL \to \mathbb{R}^m$ is orientation preserving. The space of L-adapted coframes forms a principal right $\mathrm{SO}(m)$-subbundle $P_L \subset \pi^{-1}(L) \subset P$ over L.

The equations $\eta_i = 0$ hold on P_L. Thus, by the structure equations (2.7), the relations $\beta_{ij}{\wedge}\omega_j = 0$ hold on P_L while $\omega_1{\wedge}\ldots{\wedge}\omega_m$ is nowhere vanishing. It follows from Cartan's Lemma that there are functions $h_{ijk} = h_{jik} = h_{ikj}$ on P_L so that

$$\beta_{ij} = h_{ijk}\,\omega_k\,. \tag{2.8}$$

The second fundamental form of L can then be written as

$$\mathrm{I\!I} = h_{ijk}\,J\mathbf{e}_i \otimes \omega_j\omega_k = J\mathbf{e}_i \otimes Q_i \tag{2.9}$$

where $Q_i = h_{ijk}\,\omega_j\omega_k$. The information in the second fundamental form is thus contained in the symmetric cubic form

$$C = h_{ijk}\,\omega_i\omega_j\omega_k = \omega_i\,Q_i\,, \tag{2.10}$$

which is well-defined on L. This symmetric cubic form will be referred to as the *fundamental cubic* of the special Lagrangian submanifold L.

Note that the trace of C with respect to the induced metric $g = {\omega_1}^2 + \cdots + {\omega_n}^2$ on L satisfies

$$\mathrm{tr}_g\,C = h_{iik}\,\omega_k = \beta_{ii} = -\mathrm{d}^c(\log B)\big|_L\,, \tag{2.11}$$

which is the restriction to L of an ambient 1-form. In the Calabi-Yau case, $0 = \psi_{ii} = \mathrm{i}\,\beta_{ii}$, so the fundamental cubic C is traceless.

Finally, in the flat case, the curvature vanishing condition $\mathrm{d}\psi = -\psi{\wedge}\psi$ can be separated into real and imaginary parts. The result will be referred to as the *second structure equations*:

$$\mathrm{d}\alpha_{ij} = -\alpha_{ik}\wedge\alpha_{kj} + \beta_{ik}\wedge\beta_{kj}\,, \tag{2.12a}$$

$$\mathrm{d}\beta_{ij} = -\beta_{ik}\wedge\alpha_{kj} - \alpha_{ik}\wedge\beta_{kj}\,. \tag{2.12b}$$

2.3.1. *A Bonnet-type result.* Given an m-manifold L endowed with a Riemannian metric g and a symmetric cubic form C that is traceless with respect to g, one can ask whether there is an isometric imbedding of (L, g) into $\mathbb{C}^m$ as a special Lagrangian submanifold that induces C as the fundamental cubic.

It is easy to see that there is a Bonnet-style theorem derivable from the above structure equations. Namely, there are analogs of the Gauss and Codazzi equations that give necessary and sufficient conditions for the solution of this problem.

To see this, first choose a g-orthonormal coframing $\omega = (\omega_i)$ on an open subset $U \subset L$. Then define $\eta_i = 0$ and let $\alpha_{ij} = -\alpha_{ji}$ be the unique 1-forms on U that satisfy the equations $\mathrm{d}\omega_i = -\alpha_{ij}{\wedge}\omega_j$. (The existence and uniqueness of such α is just the Fundamental Lemma of Riemannian geometry.) Write $C = h_{ijk}\,\omega_i\omega_j\omega_k$ and set $\beta_{ij} = h_{ijk}\,\omega_k$.

The equation (2.12b) then expresses the fact that C must satisfy a Codazzi-type equation with respect to g, namely, that the covariant derivative of C with respect to the Levi-Civita connection of g is fully symmetric. The equation (2.12a) then expresses the fact that that C must satisfy a Gauss-type equation with respect to g, namely, that C satisfies an algebraic equation of the form $Q_g(C) = \mathrm{Riem}(g)$, where Q_g is a certain quadratic mapping (depending on g) from symmetric cubic forms into tensors of the same algebraic type as the Riemann curvature.

Thus, when L is simply connected, these Codazzi and Gauss equations are the necessary and sufficient conditions on g and C for there to be an isometric immersion of (L, g) into $\mathbb{C}^m$ as a special Lagrangian submanifold inducing C as its fundamental cubic. Moreover, such an isometric immersion will be unique up to rigid motion.

3. Second Order Families

3.1. **The second fundamental form as a cubic.** It was already explained in §2 how the second fundamental form of a special Lagrangian submanifold $L \subset \mathbb{C}^m$ can be regarded as a symmetric cubic form that is traceless with respect to the first fundamental form.

Thus, the polynomial second order invariants of such a submanifold correspond to the $\mathrm{SO}(m)$-invariant polynomials on the space $\mathcal{H}_3(\mathbb{R}^m)$ of harmonic polynomials on $\mathbb{R}^m$ that are homogeneous of degree 3. Moreover, the $\mathrm{SO}(m)$-stabilizer of a given cubic in this space corresponds to the ambiguity in the choice of an adapted coframe for the corresponding special Lagrangian submanifold. In particular, points on special Lagrangian submanifolds at which the $\mathrm{SO}(m)$-stabilizer of the second fundamental form is nontrivial can be regarded as analogs of umbilic points in the classical theory of surfaces in 3-space.

The space $\mathcal{H}_3(\mathbb{R}^m)$ is an irreducible $\mathrm{SO}(m)$-module when $\mathrm{SO}(m)$ acts in the natural way by pullback. Thus, there are no invariant linear functions and, up to multiples, exactly one invariant quadratic polynomial, which is, essentially, the squared norm of the polynomial. It is not difficult to show that there are no invariant cubic polynomials on this space, that the space of invariant quartics is of dimension 2 (one of which is the square of the invariant quadratic), and so on. The exact structure of the ring of invariants for general m is complicated, however, and I will not discuss this further.

Since I will only be using the results of the case $m = 3$, I am going to be assuming this from now on. The space $\mathcal{H}_3(\mathbb{R}^3)$ has dimension 7, and one would expect that the 'generic' $\mathrm{SO}(3)$-orbit in this vector space to have dimension 3. In fact, I am now going to determine the orbits that have non-trivial stabilizers.

3.1.1. *Special orbits.* The main goal of this section is to prove the following result, which is undoubtedly classical even though I have been unable to locate a proof in the literature.

Proposition 1. *The* $\mathrm{SO}(3)$*-stabilizer of* $h \in \mathcal{H}_3(\mathbb{R}^3)$ *is nontrivial if and only if* h *lies on the* $\mathrm{SO}(3)$*-orbit of exactly one of the following polynomials*

1. $0 \in \mathcal{H}_3(\mathbb{R}^3)$*, whose stabilizer is* $\mathrm{SO}(3)$*.*
2. $r\,(2z^3 - 3zx^2 - 3zy^2)$ *for some* $r > 0$*, whose stabilizer is* $\mathrm{SO}(2)$*.*
3. $6s\,xyz$ *for some* $s > 0$*, whose stabilizer is the subgroup* $A_4 \subset \mathrm{SO}(3)$ *of order* 12 *generated by the rotations by an angle of* π *about the* x*-,* y*-, and* z*-axes and by rotation by an angle of* $\frac{2}{3}\pi$ *about the line* $x = y = z$*.*
4. $s\,(x^3 - 3xy^2)$ *for some* $s > 0$*, whose stabilizer is the subgroup* $S_3 \subset \mathrm{SO}(3)$ *of order* 6 *generated by the rotation by an angle of* π *about the* x*-axis and the rotation by an angle of* $\frac{2}{3}\pi$ *about the* z*-axis.*
5. $r\,(2z^3 - 3zx^2 - 3zy^2) + 6s\,xyz$ *for some* $r, s > 0$ *satisfying* $s \neq r$*, whose stabilizer is the* $\mathbb{Z}_2$*-subgroup of* $\mathrm{SO}(3)$ *generated by rotation by an angle of* π *about the* z*-axis.*
6. $r\,(2z^3 - 3zx^2 - 3zy^2) + s\,(x^3 - 3xy^2)$ *for some* $r, s > 0$ *satisfying* $s \neq r\sqrt{2}$*, whose stabilizer is the* $\mathbb{Z}_3$*-subgroup of* $\mathrm{SO}(3)$ *generated by rotation by an angle of* $\frac{2}{3}\pi$ *about the* z*-axis.*

Remark 2 (Special Values). The reader may wonder about the conditions $s \neq r$ and $s \neq r\sqrt{2}$ in the last two cases. It is not difficult to verify that the polynomial

$(2z^3 - 3zx^2 - 3zy^2) + 6xyz$ lies on the SO(3)-orbit of $2(x^3 - 3xy^2)$ and that the polynomial $(2z^3 - 3zx^2 - 3zy^2) + \sqrt{2}\,(x^3 - 3xy^2)$ lies on the SO(3)-orbit of $6\sqrt{3}\,xyz$.

Proof. Suppose that $h \in \mathcal{H}_3(\mathbb{R}^3)$ has a nontrivial stabilizer $G \subset \mathrm{SO}(3)$. Obviously $G = \mathrm{SO}(3)$ if and only if $h = 0$, so suppose that $h \neq 0$ from now on. Since G is closed in SO(3), it is compact and has a finite number of components.

Suppose first that G is not discrete. Then the identity component of G must be a closed 1-dimensional subgroup and hence conjugate to the subgroup $\mathrm{SO}(2) \subset \mathrm{SO}(3)$ consisting of the rotations about the z-axis. Thus, h lies on the orbit of a cubic polynomial that is invariant under this rotation group. By replacing h by such an element, it can be supposed that the identity component of G is SO(2).

Consider the following four subspaces of $\mathcal{H}_3(\mathbb{R}^3)$: Let V_0 be the 1-dimensional space spanned by $z(2z^2{-}3x^2{-}3y^2)$; let V_1 be the 2-dimensional space spanned by $x(4z^2{-}x^2{-}y^2)$ and $y(4z^2{-}x^2{-}y^2)$; let V_2 be the 2-dimensional space spanned by $(x^2{-}y^2)z$ and xyz; and let V_3 be the 2-dimensional space spanned by $(x^3{-}3xy^2)$ and $(3x^2y{-}y^3)$. Each of these subspaces is preserved by the elements of SO(2). Moreover, SO(2) acts trivially on V_0, while the element $R_\alpha \in \mathrm{SO}(2)$ that represents rotation by an angle α about the z-axis, acts as rotation by the angle $k\alpha$ on the 2-dimensional space V_k for $k = 1, 2, 3$.

Obviously, the only nonzero elements of $\mathcal{H}_3(\mathbb{R}^3)$ that are fixed by SO(2) are those of the form $rz(2z^2{-}3x^2{-}3y^2)$ for some nonzero r. Moreover, since z is the unique linear factor of this polynomial, it follows that G, the stabilizer of this polynomial, must preserve the z-axis. Also, since this polynomial is positive on exactly one of the two rays in the z-axis emanating from the origin, it follows that G must also fix the orientation of the z-axis. Thus, $G = \mathrm{SO}(2)$. Moreover, note that by a rotation that reverses the z-axis, the element $rz(2z^2{-}3x^2{-}3y^2)$ is carried into the element $-rz(2z^2{-}3x^2{-}3y^2)$. Thus, one can assume that $r > 0$.

Now suppose that G is discrete (and hence finite). Let $A \in G$ be an element of finite order $p > 1$. Then A is rotation about a line by an angle of the form $(2q/p)\pi$ for some integer q relatively prime to p and satisfying $0 < q < p$. Replacing h by an element in its SO(3)-orbit, I can assume that the fixed line of A is the z-axis. Since the action of A on V_k is a rotation by the angle $(2kq/p)\pi$ for $k = 1, 2, 3$, it follows that, unless either $2q/p$ or $3q/p$ are integers, then the only elements of $\mathcal{H}_3(\mathbb{R}^3)$ that are fixed by A are the elements of V_0. Since these elements have a continuous symmetry group, and so, by hypothesis, cannot be h, it follows that either $2q/p$ or $3q/p$ are integers, i.e., that $p = 2$ or $p = 3$.

If $p = 2$, then h must lie in $V_0 + V_2$, i.e., there must be constants r, s, and t, so that

$$h = r\,z(2z^2 - 3x^2 - 3y^2) + 3\bigl(s\,(2xy) + t\,(x^2{-}y^2)\bigr)\,z\,.$$

By a rotation that reverses the z-axis, if necessary, I can assume that $r \geq 0$ and then, by applying a rotation in SO(2), I can assume that $t = 0$ and $s \geq 0$. Since G is discrete, s cannot be zero, so $s > 0$. Note that A is a rotation by an angle of π about the z axis, and that this certainly preserves any h in the above form. Note also that every such h has a linear factor. In particular, to each element A of order 2 in G, there corresponds a linear factor of h that is fixed (up to a sign) by A.

If $r = 0$, then $h = 6s\,xyz$, and it is clear that the elements of G must permute the planes $x = 0$, $y = 0$, and $z = 0$. It follows that G must be the group A_4 of order 12 described in the proposition.

If $r > 0$, then it is still true that h has a linear factor, i.e.,

$$h = \big(2r\,z^2 - 3r\,x^2 - 3r\,y^2 + 6s\,xy\big)\,z.$$

When $r \neq s$, the quadratic factor in the above expression is irreducible (since r and s are positive), so G must stabilize the z-axis. In fact, since h is positive on the positive ray of the z-axis, G must actually be a subgroup of $\mathrm{SO}(2)$. Since $s > 0$, G must therefore be isomorphic to $\mathbb{Z}_2$, generated by the rotation by π about the z-axis. On the other hand, when $r = s$, the polynomial h factors as

$$h = r(\sqrt{2}\,z - \sqrt{3}\,x + \sqrt{3}\,y)(\sqrt{2}\,z + \sqrt{3}\,x - \sqrt{3}\,y)\,z\,.$$

These three linear factors of h are linearly dependent, so that h vanishes on the union of three coaxial planes that meet pairwise at an angle of $\pi/3$. Consequently, h lies on the $\mathrm{SO}(3)$-orbit of an element of the form

$$p\,(x^3 - 3xy^2) = p\,x\,\big(x - \sqrt{3}\,y\big)\big(x + \sqrt{3}\,y\big)$$

where $p > 0$. Since G must preserve these factors up to a sign, G is isomorphic to S_3 and is generated as claimed in the proposition.

Finally, assume that G has no element of order 2. Then, by the above argument, all of the nontrivial elements of G have order 3. By the well-known classification of the finite subgroups of $\mathrm{SO}(3)$, [6] it follows that G must be isomorphic to $\mathbb{Z}_3$.

Let A be a generator of G and assume (as one may, by replacing h by an element in its $\mathrm{SO}(3)$-orbit) that A is rotation by an angle of $2\pi/3$ about the z-axis. Then the elements of $\mathcal{H}_3(\mathbb{R}^3)$ that are fixed by A are the elements in $V_0 + V_3$, i.e., those of the form

$$h = r\,z(2z^2 - 3x^2 - 3y^2) + s\,(x^3 - 3xy^2) + t\,(3x^2y - y^3).$$

By a rotation about the z-axis, h can be replaced by an element in its orbit that is of the above form but that satisfies $t = 0$ and $s \geq 0$. Now, $s > 0$, since, otherwise the stabilizer of h would contain $\mathrm{SO}(2)$. After rotation by an angle of π about the x-axis if necessary, I can further assume that $r \geq 0$. In fact, $r > 0$, since, otherwise, G would be isomorphic to S_3, contrary to hypothesis.

It remains to determine those positive values of r and s (if any) for which

$$h = r\,z(2z^2 - 3x^2 - 3y^2) + s\,(x^3 - 3xy^2)$$

has a symmetry group larger than $\mathbb{Z}_3$.

If the symmetry group G is to be larger than $\mathbb{Z}_3$, then, by the aforementioned classification, either G contains an element of order 2 or G is infinite. In either case, by the above arguments, h must have a linear factor. Now, it is straightforward to verify that h has no linear factor unless $s = r\sqrt{2}$. Thus, the stabilizer is $\mathbb{Z}_3$ except in this case. On the other hand

$$z(2z^2 - 3x^2 - 3y^2) + \sqrt{2}\,(x^3 - 3xy^2) = (z + \sqrt{2}\,x)\big(x + \sqrt{3}\,y - \sqrt{2}\,z\big)\big(x - \sqrt{3}\,y - \sqrt{2}\,z\big)\,,$$

and the three linear factors vanish on three mutually orthogonal 3-planes. It follows immediately that this h lies on the orbit of $6p\,xyz$ for some $p > 0$. $\square$

The argument for Proposition 1 can be used to prove two more easy results:

[6] Up to conjugation, these subgroups consist of the cyclic subgroups, the dihedral subgroups, and the symmetry groups of the Platonic solids.

Proposition 2. *A cubic $h \in \mathcal{H}_3(\mathbb{R}^3)$ is reducible if and only if it has a symmetry of order 2. It factors into three linear factors if and only if it is either the zero cubic, has symmetry A_4, or has symmetry S_3.*

Proof. By Proposition 1, any cubic that has a symmetry of order 2 has a linear factor. Conversely, suppose that $h \in \mathcal{H}_3(\mathbb{R}^3)$ has a linear factor and is nonzero. By applying an SO(3) symmetry, it can be assumed that z divides h, implying that h has the form

$$h = z\big(r(2z^2 - 3x^2 - 3y^2) + 3p(x^2 - y^2) + 3q(2xy)\big),$$

which clearly has a symmetry of order 2 that fixes z. The quadratic factor is reducible if and only if either $r = 0$, in which case a rotation in the xy-plane reduces p to zero, so that the symmetry group is A_4, or else $p^2 + q^2 = r^2$, in which case h factors into three linearly dependent factors, so that the symmetry group is S_3. ∎

Before stating the next proposition, it will be useful to establish some notation. For any given linear function $w : \mathbb{R}^3 \to \mathbb{R}$, the subgroup $G_w \subset \mathrm{SO}(3)$ of rotations that preserve w is isomorphic to $\mathrm{SO}(2) = S^1$. The induced representation of G_w on $\mathcal{H}_3(\mathbb{R}^3) \simeq \mathbb{R}^7$ is the sum of four G_w-irreducible subspaces, V_0^w, V_1^w, V_2^w, and V_3^w, where V_0^w has dimension 1 and is the trivial representation and, for $k > 0$, V_k^w has dimension 2 and is the representation on which a rotation by an angle of α in G_w acts as a rotation by an angle of $k\alpha$.

For example, the proof of Proposition 1 lists an explicit basis for V_k^z for $0 \le k \le 3$. Note that a cubic $h \in \mathcal{H}_3(\mathbb{R}^3)$ is linear in z if and only if it lies in $V_2^z + V_3^z$. By symmetry, it follows that a cubic in $\mathcal{H}_3(\mathbb{R}^3)$ is linear in a variable w if and only if it lies in $V_2^w + V_3^w$.

Proposition 3. *The set of cubics in $\mathcal{H}_3(\mathbb{R}^3)$ that are linear in some variable is a closed semi-analytic variety of codimension 1 in $\mathcal{H}_3(\mathbb{R}^3)$ and consists of the cubics $h \in \mathcal{H}_3(\mathbb{R}^3)$ for which the projective plane curve $h = 0$ has a real singular point.*

Any cubic that is linear in two distinct variables is reducible and is on the SO(3)*-orbit of*

$$h = 3s\,xyz + sp\,(x^3 - 3xy^2),$$

which, in addition to being linear in z, is linear in $w = y + pz$ as well. When s and p are nonzero, this cubic is not linear in any other variables.

Any cubic that is linear in three distinct variables is on the SO(3)*-orbit of* $3s\,xyz$ *for some $s \ge 0$.*

Proof. A cubic h is linear in a direction w if and only if the direction generated by w is a singular point of the projectivized curve $h = 0$ in $\mathbb{RP}^2$. Thus, the first statement follows, since the set of real cubic curves with a real singular point is a semi-analytic set of codimension 1. If there are two distinct singular points, then the curve $h = 0$ must be a union of a line with a conic. If there are three distinct singular points, then the curve $h = 0$ must be the union of three nonconcurrent lines. Further details are left to the reader. ∎

3.2. **Continuous symmetry.** I now want to consider those special Lagrangian submanifolds $L \subset \mathbb{C}^3$ whose cubic second fundamental form has an SO(2) symmetry at each point.

Example 1 (SO(3)-invariant special Lagrangian submanifolds). By looking for special Lagrangian submanifolds of $\mathbb{C}^3 = \mathbb{R}^3 + \mathrm{i}\,\mathbb{R}^3$ that are invariant under the 'diagonal' action of SO(3) on the two $\mathbb{R}^3$-summands, Harvey and Lawson [12] found the following examples:

$$L_c = \left\{\, (s+\mathrm{i}\,t)\mathbf{u} \mid \mathbf{u} \in S^2 \subset \mathbb{R}^3,\ t^3 - 3s^2t = c^3 \,\right\}.$$

Here, c is a (real) constant. Note that L_0 is the union of three special Lagrangian 3-planes. When $c \neq 0$, the submanifold L_c has three components and each one is smooth and complete. In fact, these three components are isometric, as scalar multiplication in $\mathbb{C}^3$ by a nontrivial cube root of unity permutes them cyclically. Each of these components is asymptotic to one pair of 3-planes drawn from L_0. The SO(3)-stabilizer of a point of L_c is isomorphic to SO(2), so it follows that the fundamental cubic at each point has at least an SO(2)-symmetry. It is not difficult to verify that this cubic is nowhere vanishing on L_c. Note also that, for λ real and nonzero, $\lambda \cdot L_c = L_{\lambda c}$, so that, up to scaling, all of the L_c with $c \neq 0$ are isometric.

Theorem 1. *If $L \subset \mathbb{C}^3$ is a connected special Lagrangian submanifold whose cubic fundamental form has an* SO(2) *symmetry at each point, then either L is a* 3*-plane or else L is, up to rigid motion, an open subset of one of the Harvey-Lawson examples.*

Proof. Let $L \subset \mathbb{C}^3$ satisfy the hypotheses of the theorem. If the fundamental cubic C vanishes identically, then L is a 3-plane, so assume that it does not. The locus where C vanishes is a proper real-analytic subset of L, so its complement L^* is open and dense in L. Replace L by a component of L^*, so that it can be assumed that C is nowhere vanishing on L.[7]

By Proposition 1, since the stabilizer of C_x is SO(2) for all $x \in L$, there is a positive (real-analytic) function $r : L \to \mathbb{R}^+$ with the property that the equation

$$C = r\,\omega_1\left(2\,{\omega_1}^2 - 3\,{\omega_2}^2 - 3\,{\omega_3}^2\right) \tag{3.1}$$

defines an SO(2)-subbundle $F \subset P_L$ of the adapted coframe bundle $P_L \to L$. On the subbundle F, the following identities hold:

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} = \begin{pmatrix} 2r\,\omega_1 & -r\,\omega_2 & -r\,\omega_3 \\ -r\,\omega_2 & -r\,\omega_1 & 0 \\ -r\,\omega_3 & 0 & -r\,\omega_1 \end{pmatrix}. \tag{3.2}$$

Moreover, because F is an SO(2)-bundle, relations of the form

$$\begin{aligned} \alpha_{21} &= t_{21}\,\omega_1 + t_{22}\,\omega_2 + t_{23}\,\omega_3 \\ \alpha_{31} &= t_{31}\,\omega_1 + t_{32}\,\omega_2 + t_{33}\,\omega_3 \end{aligned} \tag{3.3}$$

hold on F for some functions t_{ij}. Moreover, for $i = 1, 2, 3$ there exist functions r_i on F so that

$$\mathrm{d}r = r_i\,\omega_i\,. \tag{3.4}$$

Substituting the relations (3.2), (3.3), and (3.4) into the identities

$$\mathrm{d}\beta_{ij} = -\beta_{ik} \wedge \alpha_{kj} - \alpha_{ik} \wedge \beta_{kj} \tag{3.5}$$

[7] In principle, this strategy could cause problems, but, as it is eventually going to be shown that $L = L^*$ in the general case anyway, no problems ensue.

and using the identities $\mathrm{d}\omega_i = -\alpha_{ij}{\wedge}\omega_j$ then yields polynomial relations among these quantities that can be solved, leading to relations of the form

$$\begin{aligned} \alpha_{21} &= t\,\omega_2\,, \\ \alpha_{31} &= t\,\omega_3\,, \\ \mathrm{d}r &= -4rt\,\omega_1\,, \end{aligned} \tag{3.6}$$

where, for brevity, I have written t for t_{22}.

Note that (3.6) implies that $d\omega_1 = 0$. Differentiating the last equation in (3.6) implies that there exists a function u on F so that

$$\mathrm{d}t = u\,\omega_1\,. \tag{3.7}$$

Substituting (3.6) and (3.7) into the identities

$$\mathrm{d}\alpha_{ij} = -\alpha_{ik}\wedge\alpha_{kj} + \beta_{ik}\wedge\beta_{kj} \tag{3.8}$$

and expanding, again using the identities $\mathrm{d}\omega_i = -\alpha_{ij}{\wedge}\omega_j$, yields the relations

$$u = (3r^2 - t^2), \qquad\qquad d\alpha_{23} = (t^2 + r^2)\,\omega_2\wedge\omega_3\,. \tag{3.9}$$

Differentiating these last equations yields only identities.

The structure equations found so far can be summarized as follows: $F \to L$ is an SO(2) bundle on which the 1-forms $\omega_1, \omega_2, \omega_3, \alpha_{23} (= -\alpha_{32})$ are a basis. They satisfy the structure equations

$$\begin{aligned} \mathrm{d}\omega_1 &= 0, \\ \mathrm{d}\omega_2 &= t\,\omega_1\wedge\omega_2 - \alpha_{23}\wedge\omega_3\,, \\ \mathrm{d}\omega_3 &= t\,\omega_1\wedge\omega_3 + \alpha_{23}\wedge\omega_3\,, \\ \mathrm{d}\alpha_{23} &= (t^2 + r^2)\,\omega_2\wedge\omega_3\,, \\ \mathrm{d}r &= -4rt\,\omega_1\,, \\ \mathrm{d}t &= (3r^2 - t^2)\,\omega_1\,, \end{aligned} \tag{3.10}$$

and the exterior derivatives of these equations are identities.

These equations imply that $\mathrm{d}\big(r^{3/2} + r^{-1/2}t^2\big) = 0$. Since L and F are connected, it follows that there is a constant $c > 0$ so that $r^{3/2} + r^{-1/2}t^2 = c^{-3/2}$. Consequently, there is a function θ that is well-defined on L that satisfies

$$r^{3/4} = c^{-3/4}\cos 3\theta, \qquad\qquad r^{-1/4}t = c^{-3/4}\sin 3\theta.$$

and the bound $|\theta| < \pi/6$. It then follows from the last two equations of (3.10) that

$$\omega_1 = c\,\frac{\mathrm{d}\theta}{(\cos 3\theta)^{4/3}}\,.$$

Moreover, setting $\eta_i = c^{-1}(\cos 3\theta)^{1/3}\,\omega_i$ for $i = 2$ and 3 yields

$$\mathrm{d}\eta_2 = -\alpha_{23}\wedge\eta_3\,, \qquad \mathrm{d}\eta_3 = \alpha_{23}\wedge\eta_2\,, \qquad \mathrm{d}\alpha_{23} = \eta_2\wedge\eta_3\,,$$

which are the structure equations of the metric of constant curvature 1 on S^2.

Conversely, if $d\sigma^2$ is the metric of constant curvature 1 on S^2, then, on the product $L = (-\pi/6, \pi/6) \times S^2$, consider the quadratic and cubic forms defined by

$$g = c^2\,\frac{\mathrm{d}\theta^2 + \cos^2 3\theta\,d\sigma^2}{(\cos 3\theta)^{8/3}} \qquad \text{and} \qquad C = c^2\,\frac{2\,\mathrm{d}\theta^3 - 3\cos^2 3\theta\,\mathrm{d}\theta\,d\sigma^2}{(\cos 3\theta)^{8/3}}.$$

The metric g is complete and the pair (g, C) satisfy the Gauss and Codazzi equations that ensure that (L, g) can be isometrically embedded as a special Lagrangian 3-fold in $\mathbb{C}^3$ inducing C as the fundamental cubic. Thus, for each value of c, there exists a corresponding special Lagrangian 3-fold that is complete and unique up to special Lagrangian isometries of $\mathbb{C}^3$.

Since the parameter c is accounted for by dilation in $\mathbb{C}^3$, it now follows that these special Lagrangian 3-folds are the Harvey-Lawson examples, as desired. Note that since these are complete and since r is nowhere vanishing, it follows that $L^* = L$ for the Harvey-Lawson examples, and hence for all examples. □

3.3. A_4 **symmetry.** Now consider those special Lagrangian submanifolds $L \subset \mathbb{C}^3$ whose fundamental cubic has an A_4-symmetry at each point. Unfortunately, I cannot begin the discussion by providing a nontrivial example.

Theorem 2. *The only special Lagrangian submanifold of $\mathbb{C}^3$ whose fundamental cubic has an A_4-symmetry at each point is a special Lagrangian 3-plane.*

Remark 3. It is interesting to compare the results of Theorems 1 and 2. The SO(3)-orbits consisting of the $h \in \mathcal{H}_3(\mathbb{R}^3)$ that have an SO(2) symmetry form a cone of dimension 3 in $\mathcal{H}_3(\mathbb{R}^3)$, while the ones with an A_4-symmetry form a cone of dimension 4 in $\mathcal{H}_3(\mathbb{R}^3)$. Thus, one might expect, based on 'equation counting', that the the condition of having all cubics have a SO(2)-symmetry would have fewer solutions than the condition of having all cubics have an A_4-symmetry. However, just the opposite is true.

Proof. Let $L \subset \mathbb{C}^3$ be a connected special Lagrangian submanifold with the property that its fundamental cubic C has an A_4-symmetry at each point. If C vanishes identically, then L is an open subset of a special Lagrangian 3-plane, so assume that it does not. Let $L^* \subset L$ be the dense open subset where C is nonzero.

By Proposition 1, since the stabilizer of C_x is A_4 for all $x \in L^*$, there is a positive (real-analytic) function $r : L \to \mathbb{R}^+$ for which the equation

$$C = 6r\,\omega_1\,\omega_2\,\omega_3 \tag{3.11}$$

defines an A_4-subbundle $F \subset P_L$ over L^* of the adapted coframe bundle $P_L \to L$. On F, the following identities hold:

$$\begin{pmatrix}\beta_{11} & \beta_{12} & \beta_{13}\\ \beta_{21} & \beta_{22} & \beta_{23}\\ \beta_{31} & \beta_{32} & \beta_{33}\end{pmatrix} = \begin{pmatrix}0 & r\,\omega_3 & r\,\omega_2\\ r\,\omega_3 & 0 & r\,\omega_1\\ r\,\omega_2 & r\,\omega_1 & 0\end{pmatrix}. \tag{3.12}$$

Since F is an A_4-bundle over L^*, there are relations

$$\begin{aligned}\alpha_{23} &= t_{11}\,\omega_1 + t_{12}\,\omega_2 + t_{13}\,\omega_3\\ \alpha_{31} &= t_{21}\,\omega_1 + t_{22}\,\omega_2 + t_{23}\,\omega_3\\ \alpha_{12} &= t_{31}\,\omega_1 + t_{32}\,\omega_2 + t_{33}\,\omega_3\end{aligned} \tag{3.13}$$

holding on F for some functions t_{ij}. Moreover, there exist functions r_i for $i = 1, 2, 3$ on F so that

$$\mathrm{d}r = r_i\,\omega_i\,. \tag{3.14}$$

Substituting the relations (3.12), (3.13), and (3.14) into the identities

$$\mathrm{d}\beta_{ij} = -\beta_{ik}\wedge\alpha_{kj} - \alpha_{ik}\wedge\beta_{kj} \tag{3.15}$$

and using the identities $\mathrm{d}\omega_i = -\alpha_{ij}\wedge\omega_j$ then yields polynomial relations among these quantities that can be solved, leading to relations

$$\alpha_{ij} = 0\,, \qquad \mathrm{d}r = 0. \tag{3.16}$$

Substituting (3.12) and (3.16) into the identities

$$\mathrm{d}\alpha_{ij} = -\alpha_{ik}\wedge\alpha_{kj} + \beta_{ik}\wedge\beta_{kj} \tag{3.17}$$

yields $r = 0$, contrary to hypothesis. □

3.4. S_3 **symmetry.** Now consider the special Lagrangian submanifolds of $\mathbb{C}^3$ whose fundamental cubic has an S_3-symmetry at every point. In contrast to the case of A_4-symmetry, there clearly are nontrivial examples of this type.

Example 2 (Products). This example is fairly trivial: Write $\mathbb{C}^3 = \mathbb{C}^1 \times \mathbb{C}^2$ and look for special Lagrangian submanifolds of the form $L = \mathbb{R} \times \Sigma$, where $\Sigma \subset \mathbb{C}^2$ is a surface. It is not difficult to see that there is a unique complex structure on $\mathbb{C}^2$ (not the given one!) with the property that L is special Lagrangian if and only if Σ is a complex curve with respect to this structure.

Explicitly, writing $z_k = x_k + \mathrm{i}\,y_k$, one sees $L = \mathbb{R} \times \Sigma$ is special Lagrangian for $\Sigma \subset \mathbb{C}^2$ if and only if the 2-forms $\mathrm{d}x_2\wedge\mathrm{d}y_2 + \mathrm{d}x_3\wedge\mathrm{d}y_3$ and $\mathrm{d}x_2\wedge\mathrm{d}y_3 + \mathrm{d}y_2\wedge\mathrm{d}x_3$ each vanish when pulled back to Σ. Since

$$(\mathrm{d}x_2\wedge\mathrm{d}y_2 + \mathrm{d}x_3\wedge\mathrm{d}y_3) + \mathrm{i}\,(\mathrm{d}x_2\wedge\mathrm{d}y_3 + \mathrm{d}y_2\wedge\mathrm{d}x_3) = (\mathrm{d}x_2 - \mathrm{i}\,\mathrm{d}x_3)\wedge(\mathrm{d}y_2 + \mathrm{i}\,\mathrm{d}y_3),$$

these 2-forms vanish on Σ if and only if Σ is a complex curve in $\mathbb{C}^2$ endowed with the complex structure for which $u = x_2 - \mathrm{i}\,x_3$ and $v = y_2 + \mathrm{i}\,y_3$ are holomorphic.

Now, each of these special Lagrangian 3-folds is easily seen to have its fundamental cubic be expressible as a cubic polynomial in a pair of 1-forms, from which it follows from Proposition 1 that the SO(3)-symmetry group of the cubic at each point is either everything (if the cubic vanishes at the given point) or else isomorphic to S_3.

Example 3 (Special Lagrangian cones). A more interesting example is to consider the special Lagrangian cones. Suppose that $\Sigma \subset S^5$ is a (possibly immersed) surface with the property that the cone $C(\Sigma) \subset \mathbb{C}^3$ is special Lagrangian. Then it is not difficult to show that the fundamental cubic of $C(\Sigma)$ has an S_3-stabilizer at those points where it is not zero. (This is because the cubic form uses only two of the directions.)

The necessary and sufficient conditions on Σ that $C(\Sigma)$ be special Lagrangian are easily stated: Let $\mathbf{u} : S^5 \to \mathbb{C}^3$ be the inclusion mapping. Define a 1-form θ on S^5 by $\theta = J\mathbf{u}\cdot\mathrm{d}\mathbf{u}$ and define a 2-form Ψ on S^5 by $\Psi = \mathbf{u} \lrcorner \mathrm{Im}(\Upsilon)$. Then $\Sigma \subset S^5$ has the property that $C(\Sigma)$ is special Lagrangian if and only if θ and Ψ vanish when pulled back to Σ. An elementary application of the Cartan-Kähler theorem [2] shows that any real-analytic curve $\gamma \subset S^5$ to which θ pulls back to be zero lies in an irreducible real-analytic surface Σ that satisfies these conditions. Thus, there are many such surfaces. (In the terminology of exterior differential systems, these surfaces depend on two functions of one variable.)

In addition, many explicit examples of such surfaces are now known. For example, in [13], a thorough study is done of the special Lagrangian cones that are invariant under a circle action. In fact, the differential equation for these surfaces admits a Bäcklund transformation and can be formulated as an integrable system.

In principle, the compact torus solutions can be described explicitly in terms of ϑ-functions via loop group constructions.

Example 4 (Twisted special Lagrangian cones). The special Lagrangian cones can be generalized somewhat, using a construction found in [6, §4].

Again, let $\mathbf{x} : \Sigma \to S^5$ be an immersion of a simply connected surface for which the cone on $\mathbf{x}(\Sigma)$ is special Lagrangian. Endow Σ with the metric and orientation that it inherits from this immersion and let $* : \Omega^p(\Sigma) \to \Omega^{2-p}(\Sigma)$ be the associated Hodge star operator. Since Σ is minimal, it follows that

$$*\mathrm{d}\big(*\mathrm{d}\mathbf{x}\big) + 2\mathbf{x} = 0. \tag{3.18}$$

Now, let $b : \Sigma \to \mathbb{R}$ be any solution to the second order, linear elliptic equation

$$*\mathrm{d}\big(*\mathrm{d}b\big) + 2b = 0. \tag{3.19}$$

(For example, b could be one of the components of $\mathbf{x}$.) Equations (3.18) and (3.19) imply that the vector-valued 1-form

$$\beta = \mathbf{x} *\mathrm{d}b - b *\mathrm{d}\mathbf{x} \tag{3.20}$$

is closed. Thus, there exists a $\mathbb{C}^3$-valued function $\mathbf{b} : \Sigma \to \mathbb{C}^3$ so that $\mathrm{d}\mathbf{b} = \beta$.

Now, consider the immersion $X : \mathbb{R} \times \Sigma \to \mathbb{C}^3$ defined by

$$X = \mathbf{b} + t\,\mathbf{x}\,. \tag{3.21}$$

Since $dX = \mathbf{x}\,(dt + *\mathrm{d}b) + t\,\mathrm{d}\mathbf{x} - b *\mathrm{d}\mathbf{x}$, it follows that X immerses $\mathbb{R} \times \Sigma$ as a special Lagrangian 3-fold in $\mathbb{C}^3$, at least away from the locus $t = b = 0$ in $\mathbb{R} \times \Sigma$, where X fails to be an immersion. Moreover, at those places where the fundamental cubic of this immersed submanifold is nonzero, it has S_3-symmetry.

It turns out [6] that the image $X(\mathbb{R} \times \Sigma)$ determines the data $\mathbf{x} : \Sigma \to S^5$ and $b : \Sigma \to \mathbb{R}$ up to a replacement of the form $(\mathbf{x}, b) \mapsto (-\mathbf{x}, -b)$, except in the case that $\mathbf{x}(\Sigma)$ lies in a special Lagrangian 3-plane, in which case, $X(\mathbb{R} \times \Sigma)$ lies in a parallel 3-plane.

Note that when $b = 0$, the function $\mathbf{b}$ is constant, so that $X(\mathbb{R} \times \Sigma)$ is just a translation of the cone on Σ. Thus, these examples properly generalize the special Lagrangian cones. I will refer to these examples as *twisted special Lagrangian cones*.

As explained in [6], this example can be generalized somewhat by allowing $\mathbf{x} : \Sigma \to S^5$ to be a branched immersion that is an integral manifold of θ and Ψ, but then one must allow b to have 'pole-type' singularities at the branch points of the immersion $\mathbf{x}$.

Theorem 3. *Suppose that $L \subset \mathbb{C}^3$ is a connected special Lagrangian 3-fold with the property that its fundamental cubic at each point has an S_3-symmetry. Then either L is congruent to a product $\mathbb{R} \times \Sigma$ as in Example 2, or else L contains a dense open set $L^* \subset L$ such that every point of L^* has a neighborhood that lies in a twisted special Lagrangian cone $X(\mathbb{R} \times \Sigma)$, as in Example 4.*

Proof. Suppose that $L \subset \mathbb{C}^3$ satisfies the hypotheses of the theorem. If the fundamental cubic C vanishes identically on L, then L is a 3-plane and there is nothing to show, so suppose that $C \not\equiv 0$. Let $L^\circ \subset L$ be the open dense subset where $C \neq 0$.

The hypothesis that C_x has S_3-symmetry at every $x \in L^\circ$ implies that there is a positive function $s : L^\circ \to \mathbb{R}$ and an S_3-subbundle $F \subset P_L$ over L° with

projection $\mathbf{x} : F \to L^\circ \subset \mathbb{C}^3$ on which the identity

$$C = s\left({\omega_2}^3 - 3\,\omega_2{\omega_3}^2\right) \tag{3.22}$$

holds. In particular, the second fundamental form of L° has the form

$$\mathrm{I\!I} = J\mathbf{e}_2 \otimes s({\omega_2}^2 - {\omega_3}^2) + J\mathbf{e}_3 \otimes s(-2\,\omega_2\omega_3), \tag{3.23}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the vector-valued functions defined by the moving frame relation $\mathrm{d}\mathbf{x} = \mathbf{e}_1\,\omega_1 + \mathbf{e}_2\,\omega_2 + \mathbf{e}_3\,\omega_3$.

It follows from (3.23) that $L \subset \mathbb{C}^3$ is an austere submanifold of dimension 3. By Theorem 4.1 of [6], it follows that either L° is locally the product of a line in $\mathbb{C}^3$ with a minimal surface Σ in the orthogonal 5-plane, or else there exists a dense open subset $L^* \subset L^\circ$ so that every point of L^* has an open neighborhood in L^* that lies in a twisted cone constructed as in Example 4 from a minimal immersion $\mathbf{x} : \Sigma \to S^5$ and an auxiliary function $b : \Sigma \to \mathbb{R}$ satisfying (3.19).

Since the group of translations and SU(3)-rotations in $\mathbb{C}^3$ acts transitively on the space of lines, it follows that if L is locally an orthogonal product $\mathbb{R} \times \Sigma$ and is special Lagrangian, then, up to translation by a constant, Σ must be a complex curve in the complex 2-plane P orthogonal to the linear factor, where the complex structure on P is taken to be as defined in Example 2.

On the other hand, if L is not locally an orthogonal product and so is a twisted cone as described above, then one sees from the formula for dX derived in Example 4 that the immersion $\mathbf{x} : \Sigma \to S^5$ must not only be minimal, but must have the property that $\mathbf{x}^*\theta = \mathbf{x}^*\Psi = 0$ as well, as desired. □

Remark 4 (Singular behavior). The reader may be annoyed by the apparent need to restrict to the open dense subset $L^* \subset L$. However, there are subtle singularity issues that seem to require this. For more discussion, see the final pages of [6].

Remark 5 (Austerity). Theorem 3 implies that the austere special Lagrangian 3-folds in $\mathbb{C}^3$ are completely described by Examples 2 and 4.

Remark 6 (Generality). The reader knowledgeable about exterior differential systems may wonder about the generality of the austere special Lagrangian 3-folds in the sense of Cartan-Kähler theory. While I have avoided this approach to the analysis of these examples in this treatment, I should confess that I first understood the local geometry of these examples by doing a Cartan-Kähler analysis. The obvious exterior differential system that one writes down for these examples is involutive, with Cartan characters $s_1 = 4$ and $s_2 = s_3 = 0$. The characteristic variety of the involutive prolongation consists of two complex conjugate points, each of multiplicity 2.

3.4.1. *Structure equations.* For use in the next section, I will record here the structure equations that one derives for systems of this kind. I will maintain the notation established in the proof of Theorem 3 for $L^\circ \subset L$, the function s, and the S_3-bundle $\pi : F \to L^\circ$. Thus, the fundamental cubic factors as

$$C = s\left({\omega_2}^3 - 3\,\omega_2{\omega_3}^2\right) = s\,\omega_2\,(\omega_2 + \sqrt{3}\,\omega_3)\,(\omega_2 - \sqrt{3}\,\omega_3). \tag{3.24}$$

In particular, the equations

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s\,\omega_2 & -s\,\omega_3 \\ 0 & -s\,\omega_3 & -s\,\omega_2 \end{pmatrix} \tag{3.25}$$

hold on F. Moreover, because F is an S_3-bundle, relations of the form

$$\begin{aligned}\alpha_{23} &= t_{11}\,\omega_1 + t_{12}\,\omega_2 + t_{13}\,\omega_3\\ \alpha_{31} &= t_{21}\,\omega_1 + t_{22}\,\omega_2 + t_{23}\,\omega_3\\ \alpha_{12} &= t_{31}\,\omega_1 + t_{32}\,\omega_2 + t_{33}\,\omega_3\end{aligned} \tag{3.26}$$

hold on F for some functions t_{ij}. Also, for $i = 1, 2, 3$ there exist functions s_i on F so that

$$\mathrm{d}s = s_i\,\omega_i\,. \tag{3.27}$$

Substituting the relations (3.25), (3.26), and (3.27) into the identities

$$\mathrm{d}\beta_{ij} = -\beta_{ik}\wedge\alpha_{kj} - \alpha_{ik}\wedge\beta_{kj} \tag{3.28}$$

and using the identities $\mathrm{d}\omega_i = -\alpha_{ij}\wedge\omega_j$ then yields polynomial relations among these quantities that can be solved, leading to relations of the form

$$\begin{aligned}\alpha_{23} &= r_2\,\omega_1 - t_2\,\omega_2 - t_3\,\omega_3\,,\\ \alpha_{31} &= -3r_2\,\omega_2 - 3r_3\,\omega_3\,,\\ \alpha_{12} &= \quad 3r_3\,\omega_2 - 3r_2\,\omega_3\,,\\ \mathrm{d}s &= 3s\big(r_3\,\omega_1 - t_3\,\omega_2 + t_2\,\omega_3\big)\,,\end{aligned} \tag{3.29}$$

where I have renamed the covariant derivative variables in the solution for simplicity and symmetry of notation.

It is worth mentioning that not all of these functions on F are invariant under the action of the group S_3 on the fibers. The functions s and r_2 are invariant, the function r_3 and the 1-form ω_1 are invariant under the odd order elements of S_3 but switch sign under an element of order 2, and the complex function $t = (t_2, t_3) : F \to \mathbb{R}^2$ is S_3-equivariant when $\mathbb{R}^2$ is appropriately identified with the nontrivial irreducible representation of dimension 2 of S_3. Thus, s, r_2, ${r_3}^2$, and ${t_2}^2 + {t_3}^2$ are all well-defined on L, but r_3, for example, is only well-defined up to a sign.

Because the exterior differential system mentioned above is involutive, it can be shown that one can prescribe the functions t_2, t_3, r_2, and r_3 essentially arbitrarily along any curve on which ${\omega_2}^2 + {\omega_3}^2$ is nonzero (the curves defined by the differential equations $\omega_2 = \omega_3 = 0$ are characteristic) and generate a solution.

The functions r_2 and r_3 vanish identically if and only if L is an orthogonal product. Otherwise, L is (locally) a twisted cone.

The structure equations derived so far imply that

$$\omega_2\wedge\mathrm{d}\omega_2 = \tfrac14\,(\omega_2 \pm \sqrt{3}\,\omega_3)\wedge\mathrm{d}(\omega_2 \pm \sqrt{3}\,\omega_3) = -2r_2\,\omega_1\wedge\omega_2\wedge\omega_3\,,$$

so it follows that the three linear factors of C define integrable 2-plane fields on L° if and only if $r_2 \equiv 0$ (and that if any one of the three is integrable, then so are the other two).

If one considers the differential system with the additional condition $r_2 \equiv 0$, one sees that it implies the structure equation $dr_3 = 3{r_3}^2\,\omega_1$ and that the reduced system, with this condition added, is still involutive, but now with Cartan characters $s_1 = 2$ and $s_2 = s_3 = 0$. In fact, the condition $r_2 = 0$ characterizes the special Lagrangian cones and (under the additional condition $r_3 = 0$) the orthogonal products.

Finally, note that, when $r_2 = 0$, the 2-dimensional leaves of the 2-plane field defined by $\omega_2 = 0$ will not lie in 3-planes unless $t_3 \equiv 0$. Since the condition $t_3 = 0$ is not S_3-invariant unless $t_2 = 0$ as well, it follows that, except in the very special

case $r_2 = t_3 = t_2 = 0$, at most one of the three foliations has its leaves lying in 3-planes.

It is not difficult to show that, up to congruence, there is only one example that satisfies $r_2 \equiv t_3 \equiv t_2 \equiv 0$, namely, the Harvey-Lawson example $L \subset \mathbb{C}^3$ defined in the standard coordinates by the equations $|z_1|^2 = |z_2|^2 = |z_3|^2$ and $\mathrm{Im}(z_1 z_2 z_3) = 0$. This cone is cut into surfaces by three distinct families of Lagrangian planes. For example, each element of the circle of Lagrangian planes defined by the relations

$$z_1 - \mathrm{e}^{\mathrm{i}\theta}\overline{z_2} = z_3 - \mathrm{e}^{-2\mathrm{i}\theta}\overline{z_3} = 0$$

meets L in a 2-dimensional cone. One gets the other two families by permuting the coordinates z_i.

In the case that only one of the three linear divisors of C defines a foliation by surfaces that lie in 3-planes, one can reduce to a $\mathbb{Z}_2$-subbundle of F by imposing the condition that $t_3 \equiv 0$. Then, by pursuing the calculation of the integrability conditions, one finds that the remaining quantities r_3 and t_2 must satisfy the equations

$$\begin{aligned} \mathrm{d}r_3 &= 3{r_3}^2\,\omega_1 \\ \mathrm{d}t_2 &= 3t_2 r_3\,\omega_1 + ({t_2}^2 + 9{r_3}^2 - 2s^2)\,\omega_3 \end{aligned} \tag{3.30}$$

Note that if r_3 vanishes anywhere, it vanishes identically. As already mentioned, this is the case of a product. It is not difficult to show that any connected example of this kind is congruent to an open subset of the special Lagrangian 3-fold L_c defined by the equations

$$y_1 = (x_2 - \mathrm{i}\,x_3)^2 - (y_2 + \mathrm{i}\,y_3)^2 - c^2 = 0,$$

where $c > 0$ is a real parameter. This meets the circle of Lagrangian planes defined by

$$y_1 = \cos\theta\,x_2 - \sin\theta\,y_2 = \cos\theta\,x_3 + \sin\theta\,y_3 = 0$$

in congruent surfaces that are hyperbolic cylinders.

On the other hand, if r_3 is nonzero, one can reduce the structure bundle to a parallelization of L by imposing the conditions $t_3 = 0$ and $r_3 > 0$, so assume this. By the structure equations, the expression $G = (s^2 + {t_2}^2 + 9\,{r_3}^2)s^{-2/3}{r_3}^{-4/3}$ is constant on L. Moreover, one easily sees from the structure equations that the vector field X that satisfies $\omega_1(X) = \omega_3(X) = 0$ and $\omega_2(X) = s^{-1/3}{r_3}^{-2/3}$ is a symmetry vector field of the system and hence must correspond to an ambient symmetry of the corresponding solution. Since this symmetry must fix the vertex of the cone, it follows that it is a rotation. Pursuing this observation, it is not difficult to show that all of these solutions can be described as follows: Let $\lambda_1 \geq \lambda_2 > 0 > \lambda_3$ be real numbers satisfying $\lambda_1 + \lambda_2 + \lambda_3 = 0$, and consider the 3-fold $L_\lambda \subset \mathbb{C}^3$ consisting of the points of the form

$$\left(r_1\,\mathrm{e}^{\mathrm{i}(\pi/6+\lambda_1\,t)}, r_2\,\mathrm{e}^{\mathrm{i}(\pi/6+\lambda_2\,t)}, r_3\,\mathrm{e}^{\mathrm{i}(\pi/6+\lambda_3\,t)}\right)$$

where t, r_1, r_2, and r_3 are real numbers satisfying $\lambda_1\,{r_1}^2 + \lambda_2\,{r_2}^2 + \lambda_3\,{r_3}^2 = 0$. Then L_λ is a special Lagrangian cone with a foliation by 2-dimensional 3-plane slices given by $\mathrm{d}t = 0$. (These 3-plane slices are all congruent and are Euclidean cones.) Moreover, every L of the type under discussion is congruent to L_λ for some λ.

Note, by the way, that L_λ is not closed unless the ratios of the λ_i are rational. Thus, for 'generic' λ, the cone L_λ is dense in the 4-dimensional cone in $\mathbb{C}^3$ defined by the equations

$$\lambda_1\,|z_1|^2 + \lambda_2\,|z_2|^2 + \lambda_3\,|z_3|^2 = \mathrm{Re}(z_1 z_2 z_3) = 0.$$

Part of the significance of these examples will be explained in the next section.

3.5. $\mathbb{Z}_2$ **symmetry.** Now consider a special Lagrangian submanifold $L \subset \mathbb{C}^3$ whose fundamental cubic C has a $\mathbb{Z}_2$-symmetry at each point. Equivalently, by Proposition 2, this is the same as assuming that the fundamental cubic C is reducible at each point.

Several nontrivial examples have already been seen: In fact, if C_x has a continuous stabilizer at each x or if C_x has an S_3-stabilizer at each x, then Proposition 1 shows that C_x must be reducible at each point. In the first case, the examples are classified by Theorem 1 and in the second case, the examples are classified by Theorem 3. However, these examples have stabilizer groups strictly larger than $\mathbb{Z}_2$, so the interesting question is whether there exist any other examples. By Proposition 1 and Theorem 2, any such example L will have to have the property that the $\mathrm{SO}(3)$-stabilizer of C_x is exactly $\mathbb{Z}_2$ for generic $x \in L$.

Before discussing explicit examples, I will describe a geometrically interesting condition that forces there to be a $\mathbb{Z}_2$-symmetry of C_x for all $x \in L$.

Proposition 4. *Let $L \subset \mathbb{C}^3$ be a special Lagrangian submanifold that supports a smooth codimension 1 foliation $\mathcal{S}$ with the property that each $\mathcal{S}$-leaf $S \subset L$ lies in a 3-plane. Then C_x is reducible for all $x \in L$. In particular, the $\mathrm{SO}(3)$-stabilizer of C_x contains an element of order 2.*

Proof. It suffices to assume that L is connected, so do this.

If any $\mathcal{S}$-leaf S is planar, even locally, then this plane must be ω-isotropic and Harvey and Lawson's Theorem 5.5 of §III in [12] implies that L itself must contain an open subset of a special Lagrangian 3-plane. By real-analyticity, it follows that L itself is planar and hence that C_x vanishes identically for all $x \in L$. Thus, from now on, I can assume that none of the $\mathcal{S}$-leaves are planar and that L itself is nonplanar.

Choose $x \in L$ and restrict L to a neighborhood U on which the foliation can be expressed a product, i.e., $U = X\big((0,1)\times D\big)$ for some open domain $D \subset \mathbb{R}^2$, and the $\mathcal{S}$-leaves in U are of the form $X(t,D)$ for $t \in (0,1)$. Then, by hypothesis, for each $t \in (0,1)$, there exists a unique real 3-plane $P(t) \subset \mathbb{C}^3$ so that $U \cap P(t) = X(t,D)$, and the surface $U \cap P(t)$ is ω-isotropic. Since the surface $U \cap P(t)$ is non-planar, the plane $P(t)$ itself must Lagrangian, although it cannot be special Lagrangian, since, otherwise, the uniqueness aspect of Harvey and Lawson's Theorem 5.5 would imply that $U \subset P(t)$, contradicting the assumption that L is not planar. It is not difficult to see that the curve $t \mapsto P(t)$ must be smooth, since the foliation $\mathcal{S}$ is assumed to be smooth.

Now, consider the $\mathrm{SO}(2)$-subbundle $F \subset P_L$ over U with the property that the vector-valued functions $\mathbf{e}_2$ and $\mathbf{e}_3$ are an oriented basis of the tangent space to the $\mathcal{S}$-leaves. Then ω_1 is well-defined on U and vanishes when pulled back to any $\mathcal{S}$-leaf.

Now, the set of Lagrangian planes that contain $\mathbf{e}_2$ and $\mathbf{e}_3$ is the circle of 3-planes that contain $\mathbf{e}_2$, $\mathbf{e}_3$ and that are contained in the span of $\mathbf{e}_1$, $J\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. In particular, $P(t)$ lies in this plane for each leaf $X(t,D) \subset U$. Since each leaf $\omega_1 = 0$

lies in $P(t)$, it follows that the second fundamental form

$$\mathrm{I\!I} = J\mathbf{e}_1 \otimes Q_1 + J\mathbf{e}_2 \otimes Q_2 + J\mathbf{e}_3 \otimes Q_3$$

has the property that Q_2 and Q_3 must vanish when restricted to the 2-planes defined by $\omega_1 = 0$, i.e., it must be true that Q_2 and Q_3 are multiples of ω_1. However, by Euler's homogeneity relation

$$C = \omega_1\, Q_1 + \omega_2\, Q_2 + \omega_3\, Q_3\,,$$

it now follows that C itself must be a multiple of ω_1, i.e., C is reducible at every point of U, as desired.

Finally, by Proposition 2, the SO(3)-stabilizer of C_x must contain an element of order 2 for all $x \in L$. □

Remark 7 (Non-integrable factors and non-planar foliations). It is worth pointing out that there are examples of special Lagrangian 3-folds $L \subset \mathbb{C}^3$ for which the fundamental cubic C is reducible, but for which the factors of C do not define codimension 1 foliations of L. In fact, by the discussion in 3.4.1, it follows that, for the generic special Lagrangian 3-fold L for which the fundamental cubic C has an SO(3)-stabilizer isomorphic to S_3, the cubic C factors into three linear factors, no one of which defines an integrable 2-plane field.

Moreover, even in the case where $r_2 \equiv 0$ (in which case, L is a cone), so that the three factors are each integrable, the leaves of the three foliations will not lie in 3-planes unless $t_3 \equiv 0$, which does not hold for the general special Lagrangian cone.

Example 5 (Lawlor-Harvey). This example was first found by Lawlor [18], and was subsequently generalized and extended by Harvey [11, 7.78–9]. While their results are valid in all dimensions, I will only discuss the dimension 3 case.

They show that, for any compact 2-dimensional ellipsoid $E \subset P$ where $P \subset \mathbb{C}^3$ is a Lagrangian (but not special Lagrangian) 3-plane, the special Lagrangian extension L of E is foliated in codimension 1 by a 1-parameter family of 2-dimensional ellipsoids, each of which lies in a 3-plane. By Proposition 4, it follows that the fundamental cubic of the Lawlor-Harvey examples must be reducible at each point, and thus have a symmetry of order 2.

It is not difficult to see that, except when the ellipsoid is a round 2-sphere, the Lawlor-Harvey examples are not special cases of either the SO(2)-symmetry examples or of the S_3-symmetry examples. Thus, it follows that, at least at a generic point $x \in L$, the SO(3)-stabilizer of C_x must be isomorphic to $\mathbb{Z}_2$.

Remark 8 (Joyce's extension). Dominic Joyce has informed[8] me that, in fact, the Lawlor-Harvey foliation result continues to hold when E is any quadric surface in P, not necessarily an ellipsoid, or even a non-singular quadric.

Theorem 4. *Suppose that $L \subset \mathbb{C}^3$ is a connected special Lagrangian 3-fold whose fundamental cubic C is of $\mathbb{Z}_2$-stabilizer type on an open dense subset $L^* \subset L$. Then L^* has a codimension 1 foliation $\mathcal{S}$ such that each $\mathcal{S}$-leaf lies in a 3-plane and is, moreover, a quadric surface in that 3-plane. The space of maximally extended special Lagrangian 3-folds of this type is finite dimensional and, in fact, coincides with the space of Lawlor-Harvey examples, as extended by Joyce.*

[8] private communication, 3 July 2000

Proof. By assumption, at a generic point $x \in L$, the SO(3)-stabilizer subgroup of C_x is isomorphic to $\mathbb{Z}_2$. Let $L^\circ \subset L$ be the open, dense subset where this holds. Then by Proposition 1, there exist positive functions $r, s : L^\circ \to \mathbb{R}$ with $r \neq s$ and a $\mathbb{Z}_2$-subbundle $F \subset P_L$ over L° on which the following identity holds:

$$C = r\,\omega_1\,(2{\omega_1}^2 - 3{\omega_2}^2 - 3{\omega_3}^2) + 6s\,\omega_1\omega_2\omega_3\,. \tag{3.31}$$

(Of course, $\pi : F \to L^\circ$ is a double cover and the reader can just think of the coframing ω as being well-defined on L° up to the ambiguity of replacing ω_2 and ω_3 by $-\omega_2$ and $-\omega_3$.)

Consequently, on the subbundle F, the following identities hold:

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} = \begin{pmatrix} 2r\,\omega_1 & s\,\omega_3 - r\,\omega_2 & s\,\omega_2 - r\,\omega_3 \\ s\,\omega_3 - r\,\omega_2 & -r\,\omega_1 & s\,\omega_1 \\ s\,\omega_2 - r\,\omega_3 & s\,\omega_1 & -r\,\omega_1 \end{pmatrix}. \tag{3.32}$$

Moreover, because F is a $\mathbb{Z}_2$-bundle, relations of the form

$$\begin{aligned} \alpha_{23} &= t_{11}\,\omega_1 + t_{12}\,\omega_2 + t_{13}\,\omega_3 \\ \alpha_{31} &= t_{21}\,\omega_1 + t_{22}\,\omega_2 + t_{23}\,\omega_3 \\ \alpha_{12} &= t_{31}\,\omega_1 + t_{32}\,\omega_2 + t_{33}\,\omega_3 \end{aligned} \tag{3.33}$$

hold on F for some functions t_{ij}. Moreover, for $i = 1, 2, 3$ there exist functions r_i and s_i on F so that

$$\mathrm{d}r = r_i\,\omega_i\,, \qquad \mathrm{d}s = s_i\,\omega_i\,. \tag{3.34}$$

Substituting the relations (3.32), (3.33), and (3.34) into the identities

$$\mathrm{d}\beta_{ij} = -\beta_{ik}\wedge\alpha_{kj} - \alpha_{ik}\wedge\beta_{kj} \tag{3.35}$$

and using the identities $\mathrm{d}\omega_i = -\alpha_{ij}\wedge\omega_j$ then yields polynomial relations among these quantities that can be solved,[9] leading to relations of the form

$$\begin{aligned} \mathrm{d}r &= 2(s^2+2r^2)t_1\,\omega_1 + (2rst_3 + s^2t_2)\,\omega_2 - (2rst_2 + s^2t_3)\,\omega_3\,, \\ \mathrm{d}s &= s\big(6rt_1\,\omega_1 + (2st_3 + rt_2)\,\omega_2 - (2st_2 + rt_3)\,\omega_3\big)\,, \\ \alpha_{23} &= \tfrac{1}{2}(st_2 - rt_3)\,\omega_2 + \tfrac{1}{2}(st_3 - rt_2)\,\omega_3 \\ \alpha_{31} &= -st_2\,\omega_1 + st_1\,\omega_2 - rt_1\,\omega_3\,, \\ \alpha_{12} &= -st_3\,\omega_1 + rt_1\,\omega_2 - st_1\,\omega_3\,, \end{aligned} \tag{3.36}$$

where, for brevity, I have introduced the notation

$$t_1 = -t_{23}/r\,, \qquad t_2 = -t_{21}/s\,, \qquad t_3 = -t_{31}/s\,.$$

Using (3.36) to expand out the identities

$$0 = \mathrm{d}(\mathrm{d}\omega_1) = \mathrm{d}(\mathrm{d}\omega_2) = \mathrm{d}(\mathrm{d}\omega_3) = \mathrm{d}(\mathrm{d}r) = \mathrm{d}(\mathrm{d}s)$$

yields relations on the exterior derivatives of t_1, t_2, and t_3. These can be expressed by the condition that there exist functions u_1, u_2, and u_3 so that the equations

$$\begin{aligned} \mathrm{d}t_1 &= (s\,u_1 - 3r - 3r^2\,{t_1}^2)\,\omega_1 \\ \mathrm{d}t_2 &= -3t_1(r\,t_2 - s\,t_3)\,\omega_1 + (u_2 - \tfrac{3}{2}r\,{t_2}^2)\,\omega_2 + (u_3 + \tfrac{3}{2}s\,{t_2}^2)\,\omega_3\,, \\ \mathrm{d}t_3 &= -3t_1(r\,t_3 - s\,t_2)\,\omega_1 - (u_3 + \tfrac{3}{2}s\,{t_3}^2)\,\omega_2 - (u_2 - \tfrac{3}{2}r\,{t_3}^2)\,\omega_3 \end{aligned} \tag{3.37}$$

[9] During the derivation of (3.36), one uses the assumptions that r, s and $r^2 - s^2$ are all nonzero.

hold. Substituting (3.36) and (3.37) into the identities

$$\mathrm{d}\alpha_{ij} = -\alpha_{ik}\wedge\alpha_{kj} + \beta_{ik}\wedge\beta_{kj} \tag{3.38}$$

and expanding, again using the identities $\mathrm{d}\omega_i = -\alpha_{ij}\wedge\omega_j$, yields

$$\begin{aligned} u_2 &= \tfrac{1}{2}\big(-2r\,{t_1}^2 + r\,{t_2}^2 - 3s\,t_2t_3 + r\,{t_3}^2\big) - r - s\,u_1\,,\\ u_3 &= \tfrac{1}{2}\big(\;\;2s\,{t_1}^2 - s\,{t_2}^2 + 3r\,t_2t_3 - s\,{t_3}^2\big) + s + r\,u_1\,. \end{aligned}\tag{3.39}$$

Finally, expanding out the identities $\mathrm{d}(\mathrm{d}t_1) = \mathrm{d}(\mathrm{d}t_2) = \mathrm{d}(\mathrm{d}t_3) = 0$ shows that they are equivalent to the formula

$$\begin{aligned} \mathrm{d}u_1 = \;& -2t_1\big(3r\,u_1 + s\,(-{t_1}^2 + 2\,{t_2}^2 + 2\,{t_3}^2)\big)\,\omega_1\\ & - \big(u_1(rt_2+st_3) + 3(rt_3+st_2)(1+{t_1}^2)\big)\,\omega_2\\ & + \big(u_1(st_2+rt_3) - 3(rt_2+st_s)(1+{t_1}^2)\big)\,\omega_3\,. \end{aligned}\tag{3.40}$$

The exterior derivative of (3.40) is an identity.

For future use, I record the formulae

$$\begin{aligned} \mathrm{d}\omega_1 &= -s\,(t_3\,\omega_2 - t_2\,\omega_3)\wedge\omega_1\,,\\ \mathrm{d}\omega_2 &= t_1\,(r\,\omega_2 - s\,\omega_3)\wedge\omega_1 + \tfrac{1}{2}(rt_3 - st_2)\,\omega_2\wedge\omega_3\,,\\ \mathrm{d}\omega_3 &= t_1\,(r\,\omega_3 - s\,\omega_2)\wedge\omega_1 + \tfrac{1}{2}(rt_2 - st_3)\,\omega_2\wedge\omega_3\,. \end{aligned}\tag{3.41}$$

which follow from the identities $\mathrm{d}\omega_i = -\alpha_{ij}\wedge\omega_j$ coupled with (3.36).

At this point, it is worthwhile taking stock of what has been accomplished. Consider the system of quantities

$$\omega_1,\,\omega_2,\,\omega_3,\,r,\,s,\,t_1,\,t_2,\,t_3,\,u_1\,.$$

The formulae (3.41), (3.36), (3.37), and (3.40) express the exterior derivatives of these quantities as polynomials in these quantities. Moreover, the relation $\mathrm{d}(\mathrm{d}q) = 0$ for q any one of these quantities follows by formal expansion and use of the given exterior derivative formulae.

By a theorem[10] of Élie Cartan, for any six constants $\bar r$, $\bar s$, $\bar t_1$, $\bar t_2$, $\bar t_3$, $\bar u_1$, there exists an open neighborhood U of $0\in\mathbb{R}^3$ that is endowed with three linearly independent 1-forms ω_i and six functions r, s, t_1, t_2, t_3, u_1 that satisfy the equations (3.41), (3.36), (3.37), and (3.40) and also satisfy

$$r(0)=\bar r,\quad s(0)=\bar s,\quad t_1(0)=\bar t_1,\quad t_2(0)=\bar t_2,\quad t_3(0)=\bar t_3,\quad u_1(0)=\bar u_1\,.$$

Moreover these functions and forms are real-analytic and unique in a neighborhood of 0, up to a real-analytic local diffeomorphism fixing 0.

Now, given such a system (ω,r,s,t,u) on a simply connected 3-manifold L, one can set $\eta_i = 0$, define $\alpha_{ij} = -\alpha_{ji}$ by the last three equations of (3.36), define $\beta_{ij} = \beta_{ji}$ by the equations (3.32), and see that the affine structure equations

$$\begin{aligned} \mathrm{d}\omega_i &= -\alpha_{ij}\wedge\omega_j + \beta_{ij}\wedge\eta_j\,,\\ \mathrm{d}\eta_i &= -\beta_{ij}\wedge\omega_j - \alpha_{ij}\wedge\eta_j\,,\\ \mathrm{d}\beta_{ij} &= -\beta_{ik}\wedge\alpha_{kj} - \alpha_{ik}\wedge\beta_{kj}\,,\\ \mathrm{d}\alpha_{ij} &= -\alpha_{ik}\wedge\alpha_{kj} + \beta_{ik}\wedge\beta_{kj} \end{aligned}\tag{3.42}$$

[10] This was originally part of Cartan's general theory of intransitive pseudo-groups. In more recent times, this theorem has been subsumed into the theory of *Lie algebroids*. For an introduction, the reader could try the Appendix of [3].

are identities. Thus, there is an immersion of L, unique up to translation and SU(3)-rotation, as a special Lagrangian 3-manifold in $\mathbb{C}^3$ that induces these structure equations.

In particular, it follows that the space of germs of special Lagrangian 3-manifolds in $\mathbb{C}^3$ whose fundamental cubics are of the form (3.31) is of dimension 6. Moreover, any two that agree to order 4 at a single point must be equal in a neighborhood. It is not difficult to argue from this that the space one gets by reducing modulo the equivalence relation defined by analytic continuation is a 3-dimensional singular space.

Now, the first of the equations (3.41) shows that the 2-plane field $\omega_1 = 0$ is integrable, moreover, the structure equations found so far imply

$$\mathrm{d}\big(\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge (J\mathbf{e}_1 - t_1\,\mathbf{e}_1)\big) \equiv 0 \mod \omega_1\,. \tag{3.43}$$

In particular, the 3-plane $\mathbf{e}_2{\wedge}\mathbf{e}_3{\wedge}(J\mathbf{e}_1 - t_1\,\mathbf{e}_1)$ is constant along each leaf of ω_1 and, moreover each such leaf lies in an affine 3-plane parallel to this 3-plane. Thus, all of these examples are foliated in codimension 1 by 3-plane sections.

Moreover, an examination of the structure equations shows that the space of congruence classes of such 3-plane sections is of dimension 3, the same as the dimension of quadric surfaces in 3-space. In fact, using the structure equations, it is not difficult to show that these 3-plane sections are, in fact, quadric surfaces. For the sake of brevity, I will not include the details of this routine calculation here.

It follows that these special Lagrangian 3-folds all belong to the class of Lawlor-Harvey examples, as extended by Joyce. □

Corollary 1. *Any connected special Lagrangian* 3*-fold* $L \subset \mathbb{C}^3$ *that is foliated in codimension* 1 *by* 3*-plane sections is an open subset of a Lawlor-Harvey-Joyce example.*

Proof. By Proposition 4, any such L must have a reducible fundamental cubic C. Thus, the SO(3)-stabilizer of C_x at each point contains a $\mathbb{Z}_2$ and so is either isomorphic to SO(2), S_3, or $\mathbb{Z}_2$.

If this stabilizer is isomorphic to SO(2) at a generic point, then Theorem 1 applies, showing that L is a Lawlor-Harvey-Joyce example.

If this stabilizer is isomorphic to S_3 at a generic point, the discussion at the end of §3.4.1 shows that the only such examples that are foliated in codimension 1 by 3-plane sections have the property that these sections are necessarily (possibly singular) quadric surfaces, so that such an L is, again, a Lawlor-Harvey-Joyce example.

Finally, if the stabilizer is isomorphic to $\mathbb{Z}_2$ at a generic point, then Theorem 4 applies. □

Remark 9 (Harvey's characterization)*.* In his proof of Theorem 7.78 in [11], Harvey gives a characterization of the Lawlor-Harvey examples that is closely related to Proposition 4. What he shows is that any special Lagrangian m-fold $L \subset \mathbb{C}^m$ that meets a certain concurrent family of Lagrangian m-planes in a codimension 1 foliation whose leaves are compact must belong to the family that they construct. When $m = 3$, Corollary 1 is more general than this, since it makes no assumption about the family of Lagrangian planes that cut L to produce the foliation and makes no assumption about compactness (or even completeness) of the leaves.

Of course, one expects that the higher dimensional analog of Corollary 1 holds, i.e., that any connected special Lagrangian m-fold $L \subset \mathbb{C}^m$ that is foliated in codimension 1 by m-plane sections is necessarily an open subset of a Lawlor-Harvey-Joyce example. I have not tried to prove this, but it should be straightforward.

3.6. $\mathbb{Z}_3$ **symmetry.** Now consider those special Lagrangian submanifolds $L \subset \mathbb{C}^3$ whose cubic second fundamental form has an $\mathbb{Z}_3$-symmetry at each point.

Example 6. Let $\Sigma \subset S^5$ be a surface such that the cone on Σ is special Lagrangian, and consider the 3-fold

$$L_\Sigma = \left\{ \, (s + \mathrm{i}\, t)\mathbf{u} \mid \mathbf{u} \in \Sigma,\ t^3 - 3s^2 t = c \, \right\},$$

where c is a (real) constant. This L_Σ is special Lagrangian. For example, see [13], where a more general result for special Lagrangian cones in $\mathbb{C}^n$ is proved.

Note that L_Σ is diffeomorphic to the disjoint union of three copies of $\mathbb{R} \times \Sigma$. In fact, each 'end' of each component of L_Σ is asymptotic to the cone on $\lambda \cdot \Sigma \subset S^5$ for some λ satisfying $\lambda^6 = 1$.

When Σ is not totally geodesic in S^5 the SO(3)-stabilizer of the fundamental cubic at a generic point of point of L_Σ is isomorphic to $\mathbb{Z}_3$.

Theorem 5. *If $L \subset \mathbb{C}^3$ is a connected special Lagrangian submanifold whose fundamental cubic has $\mathbb{Z}_3$-symmetry at each point of a dense open subset of L, then L contains a dense open set L^* such that every point of L^* has an open neighborhood in L that is an open subset of one of the special Lagrangian* 3*-folds of Example* 6*.*

Proof. Let $L \subset \mathbb{C}^3$ satisfy the hypotheses of the theorem. The locus of points $x \in L$ for which the SO(3)-stabilizer of C is larger than $\mathbb{Z}_3$ is a proper real-analytic subset of L, so its complement L^* is open and dense in L. Thus, I can, without loss of generality, replace L by a component of L^*. In other words, I can assume that the SO(3)-stabilizer of C_x is isomorphic to $\mathbb{Z}_3$ for all $x \in L$.

By Proposition 1, since the stabilizer of C_x is $\mathbb{Z}_3$ for all $x \in L$, there are positive (real-analytic) functions r and s on L with the property that the equation

$$C = r\,\omega_1\left(2\,{\omega_1}^2 - 3\,{\omega_2}^2 - 3\,{\omega_3}^2\right) + s\left({\omega_2}^3 - 3\,\omega_2{\omega_3}^2\right) \tag{3.44}$$

defines a $\mathbb{Z}_3$-subbundle $F \subset P_L$ of the adapted coframe bundle $P_L \to L$. Moreover, the expression $s - r\sqrt{2}$ is nowhere vanishing on L.

Now, on the subbundle F, the following identities hold:

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} = \begin{pmatrix} 2r\,\omega_1 & -r\,\omega_2 & -r\,\omega_3 \\ -r\,\omega_2 & -r\,\omega_1 + s\,\omega_2 & -s\,\omega_3 \\ -r\,\omega_3 & -s\,\omega_3 & -r\,\omega_1 - s\,\omega_2 \end{pmatrix}. \tag{3.45}$$

Moreover, because F is a $\mathbb{Z}_3$-bundle, relations of the form

$$\begin{aligned} \alpha_{23} &= t_{11}\,\omega_1 + t_{12}\,\omega_2 + t_{13}\,\omega_3 \\ \alpha_{31} &= t_{21}\,\omega_1 + t_{22}\,\omega_2 + t_{23}\,\omega_3 \\ \alpha_{12} &= t_{31}\,\omega_1 + t_{32}\,\omega_2 + t_{33}\,\omega_3 \end{aligned} \tag{3.46}$$

hold on F for some functions t_{ij}. Moreover, for $i = 1, 2, 3$ there exist functions r_i and s_i on F so that

$$\mathrm{d}r = r_i\,\omega_i\,, \qquad \mathrm{d}s = s_i\,\omega_i\,. \tag{3.47}$$

Substituting the relations (3.45), (3.46), and (3.47) into the identities

$$\mathrm{d}\beta_{ij} = -\beta_{ik}\wedge\alpha_{kj} - \alpha_{ik}\wedge\beta_{kj} \tag{3.48}$$

and using the identities $\mathrm{d}\omega_i = -\alpha_{ij}\wedge\omega_j$ then yields polynomial relations among these quantities that can be solved,[11] leading to relations of the form

$$\begin{aligned}
\mathrm{d}r &= -4rt_1\,\omega_1\,,\\
\mathrm{d}s &= -s\big(t_1\,\omega_1 + 3t_3\,\omega_2 - 3t_2\,\omega_3\big)\,,\\
\alpha_{23} &= -t_2\,\omega_2 - t_3\,\omega_3\\
\alpha_{31} &= t_1\,\omega_3\,,\\
\alpha_{12} &= -t_1\,\omega_2\,,
\end{aligned} \tag{3.49}$$

where, for brevity, I have introduced the notation

$$t_1 = t_{23}\,,\qquad t_2 = -t_{12}\,,\qquad t_3 = -t_{13}\,.$$

Using (3.49) to expand out the identities

$$0 = \mathrm{d}(\mathrm{d}\omega_1) = \mathrm{d}(\mathrm{d}\omega_2) = \mathrm{d}(\mathrm{d}\omega_3) = \mathrm{d}(\mathrm{d}r) = \mathrm{d}(\mathrm{d}s)$$

and also the identities

$$\mathrm{d}\alpha_{ij} = -\alpha_{ik}\wedge\alpha_{kj} + \beta_{ik}\wedge\beta_{kj}$$

yields relations on the exterior derivatives of t_1, t_2, and t_3. When these are solved, one finds that there are functions u_2 and u_3 so that the equations

$$\begin{aligned}
\mathrm{d}t_1 &= (3r^2 - {t_1}^2)\,\omega_1\\
\mathrm{d}t_2 &= -t_1t_2\,\omega_1 + u_2\,\omega_2 + (u_3+v)\,\omega_3\,,\\
\mathrm{d}t_3 &= -t_1t_3\,\omega_1 - u_2\,\omega_3 + (u_3-v)\,\omega_2
\end{aligned} \tag{3.50}$$

hold where

$$v = s^2 - \tfrac12(r^2 + {t_1}^2 + {t_2}^2 + {t_3}^2).$$

Observe that, if one sets $r = 0$ in the current structure equations, then these become, up to a trivial change of notation, the same structure equations as those for the special Lagrangian cones discussed in §3.4. This is a first hint that these examples must be related to the special Lagrangian cones.[12]

The next observation is that the structure equations

$$\mathrm{d}r = -4rt_1\,\omega_1\,,\qquad\text{and}\qquad \mathrm{d}t_1 = (3r^2 - {t_1}^2)\,\omega_1 \tag{3.51}$$

are identical (after replacing t_1 by t) to the last two equations of (3.10). In particular, there must exist a constant $c > 0$ and a function θ on L satisfying the bound $|\theta| < \pi/6$ so that

$$r^{3/4} = c^{3/4}\cos 3\theta,\qquad\qquad r^{-1/4}t_1 = c^{3/4}\sin 3\theta. \tag{3.52}$$

It then follows from (3.51) that

$$\omega_1 = \frac{\mathrm{d}\theta}{c(\cos 3\theta)^{4/3}}\,. \tag{3.53}$$

---

[11] During the derivation of (3.49), one uses the assumptions that r and s are nonzero but not the assumption that $s - r\sqrt{2}$ is nonzero.

[12] Also, if one now computes the Cartan characters of the naïve exterior differential system that models these structure equations, one finds that $s_1 = 2$ while $s_2 = s_3 = 0$ and that this exterior differential system is involutive. The characteristic variety is a pair of complex conjugate points, each of multiplicity 1.

By dilation in $\mathbb{C}^3$, one can reduce to the case $c = 1$, so assume this from now on.

Consider the following expressions:

$$
\begin{aligned}
p &= r^{-1/4}s,\\
q_2 &= r^{-1/4}t_2, \quad q_3 = r^{-1/4}t_3,\\
v_2 &= r^{-1/2}u_2, \quad v_3 = r^{-1/2}u_3,\\
\eta_2 &= r^{1/4}\omega_2, \quad \eta_3 = r^{1/4}\omega_3\,.
\end{aligned} \tag{3.54}
$$

The structure equations derived above show that

$$
\begin{aligned}
\mathrm{d}\eta_2 &= q_2\,\eta_2\wedge\eta_3\\
\mathrm{d}\eta_3 &= q_3\,\eta_2\wedge\eta_3\\
\mathrm{d}p &= -3p\,(q_3\,\eta_2 - q_2\,\eta_3)\\
\mathrm{d}q_2 &= \quad v_2\,\eta_2 + (v_3+w)\,\eta_3\\
\mathrm{d}q_3 &= -v_2\,\eta_3 + (v_3-w)\,\eta_2
\end{aligned} \tag{3.55}
$$

where $w = \frac{1}{2}(1+{q_2}^2+{q_3}^2) - p^2$.

In particular, $d(p^{1/3}\eta_2) = d(p^{1/3}\eta_3) = 0$. Let $x \in L$ be fixed and let $U \subset L$ be an x-neighborhood on which there exist functions y_2 and y_3 vanishing at x that satisfy $p^{1/3}\eta_2 = \mathrm{d}y_2$ and $p^{1/3}\eta_3 = \mathrm{d}y_3$. Then the functions (θ, y_2, y_3) are independent on U and, by shrinking U if necessary, I can assume that $(\theta, y_2, y_3)(U) \subset \mathbb{R}^3$ is a product open set of the form $I \times D$ where $I \subset (-\frac{\pi}{6}, \frac{\pi}{6})$ is a connected interval and $D \subset \mathbb{R}^2$ is a disc centered on the origin. Of course, the functions p, q_2, q_3, v_2 and v_3 can be regarded as functions on D, since their differentials are linear combinations of $\mathrm{d}y_2$ and $\mathrm{d}y_3$. In fact, these functions and forms can now be regarded as defined on the open set $(-\frac{\pi}{6}, \frac{\pi}{6}) \times D$ by simply reading the formulae above backwards. Thus, for example

$$
s = r^{1/4}p = (\cos 3\theta)^{1/4}p
$$

and so forth. This gives quantities ω_i, r, s, t_i, and u_i that are well-defined on all of $(-\frac{\pi}{6}, \frac{\pi}{6}) \times D$ and that satisfy the originally derived structure equations. It follows that there is an immersion of $(-\frac{\pi}{6}, \frac{\pi}{6}) \times D$ into $\mathbb{C}^3$ as a special Lagrangian 3-fold that extends U and pulls back the constructed forms and quantities to agree with the given ones on U. The chief difference is that each of the θ-curves in $(-\frac{\pi}{6}, \frac{\pi}{6}) \times D$ is mapped to a complete curve in $\mathbb{C}^3$.

Next, observe that the equations

$$
\left.
\begin{aligned}
\mathrm{d}\mathbf{x} &\equiv \mathbf{e}_1\,\omega_1\\
\mathrm{d}\mathbf{e}_1 &\equiv J\mathbf{e}_1\,(2r\omega_1)\\
\mathrm{d}(J\mathbf{e}_1) &\equiv -\mathbf{e}_1\,(2r\omega_1)
\end{aligned}
\right\} \quad \text{mod } \omega_2\,, \omega_3\,, \tag{3.56}
$$

which are identical to the corresponding equations in §3.2, then show that the leaves of the curve foliation defined by $\omega_2 = \omega_3 = 0$ are congruent to the leaves of the corresponding foliation by the $\mathbf{e}_1$-curves in §3.2.

Finally, note that, setting $\theta = 0$ (i.e., $t_1 = 0$ and $r = 1$ in the above structure equations on $(-\frac{\pi}{6}, \frac{\pi}{6}) \times D$ gives an immersion of D into $S^5 \subset \mathbb{C}^3$ with the property that the cone on the image Σ is a special Lagrangian 3-fold. Because the θ-curves meet this surface orthogonally, it follows easily that the image of $(-\frac{\pi}{6}, \frac{\pi}{6}) \times D$ is exactly L_Σ as described in Example 6. Further details are left to the reader. $\square$

3.7. **The ruled family.** In this last subsection, I am going to consider the generality of the set of ruled special Lagrangian 3-manifolds.

Examples of ruled special Lagrangian 3-folds in $\mathbb{C}^3$ were constructed in Harvey and Lawson's original paper [12]. These included products, special Lagrangian cones, and conormal bundles of minimal surfaces in $\mathbb{R}^3$. All of these families depend on two functions of one variable in the sense of exterior differential systems.

Harvey and Lawson also showed in [12, Theorems 4.9, 4.13] how one could deform the conormal bundle of a minimal surface in $\mathbb{R}^3$ according to the data of a harmonic function on such a surface and obtain more general ruled special Lagrangian 3-folds. (Borisenko [1] later gave a somewhat different description of the same family.) These examples depend on four functions of one variable in the sense of exterior differential systems.

On the other hand, the construction in Example 4 of twisted special Lagrangian cones provides another family of examples of ruled special Lagrangian 3-folds, again depending on four functions of one variable in the sense of exterior differential systems. It is easy to see that this family is distinct from the family described in [12, Theorems 4.9].

In this section, I am going to show that the ruled special Lagrangian 3-folds depend on *six* functions of one variable in the sense of exterior differential systems. Thus, the 'explicit' families that have been constructed so far are only a small part of the complete family. For a different description of ruled special Lagrangian 3-folds, one should consult Joyce's recent article [16].

3.7.1. *Almost CR-structures and Levi-flatness.* For the description I plan to give of the ruled special Lagrangian submanifold of $\mathbb{C}^3$, I will need some facts about a generalized notion of 'pseudo-holomorphic curves'.

Recall that an *almost CR-structure* on a manifold M is a subbundle $E \subset TM$ of even dimension equipped with a complex structure map $J : E \to E$. The *rank* of the CR-structure is the rank of E as a complex bundle and the *codimension* of the CR-structure is the rank of the quotient bundle TM/E. A (real) curve $C \subset M$ is said to be an *E-curve* if its tangent line at each point lies in E. A (real) surface $S \subset M$ is said to be *E-holomorphic* if its tangent plane at each point is a complex line in E. (In order to avoid confusion, I will not adopt the standard practice of calling these surfaces 'pseudo-holomorphic curves', or, indeed, curves of any kind.)

An almost CR-structure (E, J) will be said to be *Levi-flat* if, for any 1-form ρ on M that vanishes on E, the 2-form $\mathrm{d}\rho$ vanishes on all the 2-planes that are complex lines in E. Note that Levi-flatness is automatic when the codimension of the CR-structure is zero and that Levi-flatness generally has no implications about the 'integrability' of the almost CR-structure to a CR-structure, which is a different condition altogether.

Proposition 5. *Let (E, J) be a real-analytic, Levi-flat, almost CR-structure on M and let $C \subset M$ be a real-analytic E-curve. Then there is an E-holomorphic surface $S \subset M$ that contains C. This surface is locally unique in the sense that, for any two such surfaces S_1 and S_2, the intersection $S_1 \cap S_2$ is also an E-holomorphic surface that contains C.*

Proof. This is a straightforward application of the Cartan-Kähler Theorem [2, Chapter III] so I will only give the barest details. This is a local result, so it suffices to give a local proof.

Let r be the rank of (E, J) and let q be its codimension. For any point $x \in M$, there is an open x-neighborhood $U \subset M$ on which there exist real-analytic 1-forms $\theta_1, \ldots, \theta_q$ with real values and $\omega_1, \ldots, \omega_r$ with complex values with the property that the equations $\theta_1 = \cdots = \theta_q = 0$ define the restriction of E to U and with the property that $\omega_1, \ldots, \omega_r$ are complex linear on E and are linearly independent over $\mathbb{C}$ at each point of U. There are identities of the form

$$\mathrm{d}\theta_\alpha \equiv K_{\alpha ij}\,\omega_i \wedge \omega_j + L_{\alpha ij}\,\omega_i \wedge \overline{\omega_j} + \overline{K_{\alpha ij}}\,\overline{\omega_i} \wedge \overline{\omega_j} \mod \theta_1, \ldots, \theta_q\,.$$

The hypothesis of Levi-flatness is simply that the functions $L_{\alpha ij}$ all vanish identically. Under this hypothesis, the real-analytic exterior differential system $\mathcal{I}$ generated algebraically by the θ_α and the real and imaginary parts of the 2-forms $\omega_i \wedge \omega_j$ is involutive and each of the 1-dimensional integral elements is regular and lies in a unique 2-dimensional integral element. Now apply the Cartan-Kähler theorem. $\square$

3.7.2. *Oriented lines.* Since a ruled 3-manifold in $\mathbb{C}^3$ can be regarded as a surface in the space of lines in $\mathbb{C}^3$, it is useful to consider the geometry of this space. It is slightly more convenient to consider the space Λ of oriented lines in $\mathbb{C}^3$, so I will do this.

The space Λ is naturally diffeomorphic to the tangent bundle of S^5. Explicitly, the pair $(\mathbf{u}, \mathbf{v}) \in TS^5$ consisting of a unit vector $\mathbf{u} \in S^5$ and a vector $\mathbf{v} \in \mathbf{u}^\perp$ corresponds to the oriented line with oriented direction $\mathbf{u}$ that passes through $\mathbf{v}$. Naturally, I will regard $\mathbf{u} : \Lambda \to S^5$ and $\mathbf{v} : \Lambda \to \mathbb{C}^3$ as vector-valued functions on Λ.

Thus, a curve $\gamma : (a, b) \to \Lambda$ can be written as $\gamma(s) = \bigl(\mathbf{u}(s), \mathbf{v}(s)\bigr)$ where the curve $\mathbf{u} : (a, b) \to S^5$ and the curve $\mathbf{v} : (a, b) \to \mathbb{C}^3$ satisfy $\mathbf{u}(s) \cdot \mathbf{v}(s) = 0$ for all $s \in (a, b)$. Such a curve gives rise to a mapping $\Gamma : (a, b) \times \mathbb{R} \to \mathbb{C}^3$ by the formula

$$\Gamma(s, t) = \mathbf{v}(s) + t\,\mathbf{u}(s).$$

Assuming that γ is smooth (resp., real-analytic) then Γ is also smooth (resp., real-analytic) and Γ will be an immersion except on the locus consisting of those $(s, t) \in (a, b) \times \mathbb{R}$ where $\bigl(\mathbf{v}'(s) + t\,\mathbf{u}'(s)\bigr) \wedge \mathbf{u}(s) = 0$. On the locus where it is an immersion, the image of Γ is then a ruled surface in $\mathbb{C}^3$.

More generally, given any smooth (resp., real-analytic) map $\gamma : P \to \Lambda$ where P is a smooth (resp., real-analytic) manifold there is an induced smooth (resp., real-analytic) map $\Gamma : P \times \mathbb{R} \to \mathbb{C}^3$ defined by the same formula as above. With the appropriate 'generic' assumptions on γ, the mapping Γ will be an immersion on some open subset of $P \times \mathbb{R}$ and its image will be a ruled immersion.

There are two natural differential forms on Λ that are invariant under the complex isometries of $\mathbb{C}^3$. These are the pair of 1-forms

$$\theta = J\mathbf{u} \cdot \mathrm{d}\mathbf{u}, \qquad \text{and} \qquad \tau = J\mathbf{u} \cdot \mathrm{d}\mathbf{v}\,.$$

It is easy to see that θ and τ are linearly independent, so their common kernel $E \subset T\Lambda$ is a bundle of rank 8. The significance of these two 1-forms is revealed in the following result.

Proposition 6. *A curve $\gamma : (a, b) \to \Lambda$ is tangent to E everywhere if and only if the corresponding ruled 'surface' $\Gamma : (a, b) \times \mathbb{R} \to \mathbb{C}^3$ is ω-isotropic.*

Proof. This is immediate from the formulae for Γ and ω. $\square$

Theorem 6. *There is a complex structure J on $E \subset T\Lambda$ with the properties*

1. (E, J) *is a real-analytic, Levi-flat almost CR-structure on* Λ *that is invariant under the complex isometries of* $\mathbb{C}^3$.
2. *Any ruled special Lagrangian* 3*-fold* L *is locally the image of the* Γ *associated to an* E*-holomorphic surface* $\gamma : S \to \Lambda$. *When* L *is not a* 3*-plane, this local representation is either unique or admits at most one other such representation.*
3. *For each* E*-holomorphic surface* $\gamma : S \to \Lambda$, *the corresponding map* $\Gamma : S \times \mathbb{R} \to \mathbb{C}^3$ *is ruled and a special Lagrangian immersion on a dense open subset of* $S \times \mathbb{R}$.
4. *Any non-planar special Lagrangian* 3*-fold* L *that has two distinct rulings is a Lawlor-Harvey-Joyce example for which the* 2*-dimensional* 3*-plane sections are quadrics that are doubly ruled.*

Before going on to the proof of this result, let me state some immediate corollaries:

Corollary 2. *A connected special Lagrangian* 3*-fold* $L \subset \mathbb{C}^3$ *is ruled if and only if the set* Λ_L *of lines that intersect* L *in nontrivial open intervals* (*which is an analytic subset of* Λ) *has dimension at least* 1.

Proof. I will only sketch the proof, since the details are straightforward. First, the easy direction: If L is ruled, then the analytic set Λ_L must have dimension 2 at least.

Conversely, if the dimension of Λ_L is at least 1, then it contains an immersed analytic arc $\gamma : (a, b) \to \Lambda$, which generates a ruled surface $\Gamma(D) \subset L$ for some appropriate domain $D \subset (a, b) \times \mathbb{R}$. The surface $\Gamma(D)$ must be ω-isotropic since L is Lagrangian. Thus, the arc γ must be an E-curve. By Item 1 of Theorem 6 and Proposition 5, this arc lies in an E-holomorphic surface $\psi : S \subset \Lambda$. By Item 3 of Theorem 6, there is a dense open region $R \subset S \times \mathbb{R}$ so that $\Psi(R)$ is an immersed ruled special Lagrangian 3-fold. It is not hard to see that this $\Psi(R)$ contains at least an open subset of $\Gamma(D)$. Since by Harvey and Lawson's Theorem 5.5, the real-analytic ω-isotropic surface $\Gamma(D)$ lies in a locally unique special Lagrangian 3-fold, it follows that $\Psi(R)$ and L must intersect in an open set. Thus L is ruled on an open set. By real-analyticity and connectedness, it must be ruled everywhere. □

Corollary 3. *The ruled special Lagrangian* 3*-folds in* $\mathbb{C}^3$ *depend on six functions of one variable.*

Proof. Combine Theorem 6 and Proposition 5. □

Remark 10 (The characteristic variety). The characteristic variety of this system turns out to be a pair of complex conjugate points, each of multiplicity 3. This is particularly interesting for the following reason: The condition that the fundamental cubic at each point be singular is a single equation of second order on the special Lagrangian 3-fold. Now, as usual, a Lagrangian manifold can be written as a gradient graph of a potential function, in which case, the special Lagrangian condition is a single second order elliptic equation for the potential. Then the condition that the fundamental cubic be singular is a single *third* order equation for the potential. By the general theory, the characteristic variety of a system consisting of a single elliptic second order equation and a single third order equation consists of at most

six points by Bezout's Theorem. Remarkably, the 'singular cubic' system turns out to have such a 'maximal' characteristic variety and to be involutive.

This must be quite rare. In fact, so far, I have been unable to find another example of a single pointwise equation on the second fundamental form that is involutive and has six points in its characteristic variety.

Now for the proof of Theorem 6.

Proof. First, I will define the almost CR-structure on Λ and show that it is Levi-flat. Consider the mapping $\lambda : F \to \Lambda$ that sends the coframe $u : T_x \to \mathbb{C}^3$ to the oriented line spanned by $\mathbf{e}_1(u)$ that passes through x. Since the structure equations give

$$\left.\begin{aligned} \mathrm{d}\mathbf{x} &\equiv \mathbf{e}_2\,\omega_2 + \mathbf{e}_3\,\omega_3 + J\mathbf{e}_1\,\eta_1 + J\mathbf{e}_2\,\eta_2 + J\mathbf{e}_3\,\eta_3 \\ \mathrm{d}\mathbf{e}_1 &\equiv \mathbf{e}_2\,\alpha_{21} + \mathbf{e}_3\,\alpha_{31} + J\mathbf{e}_1\,\beta_{11} + J\mathbf{e}_2\,\beta_{21} + J\mathbf{e}_3\,\beta_{31} \end{aligned}\right\} \mod \mathbf{e}_1\,, \tag{3.57}$$

it follows that the ten 1-forms that appear on the right-hand side of this equation are λ-semibasic and it is evident that $\lambda^*(\theta) = \beta_{11}$ while $\lambda^*(\tau) = \eta_1$. The fibers of λ are cosets of the subgroup of the motion group that fixes an oriented line in $\mathbb{C}^3$ and hence are diffeomorphic to $\mathbb{R} \times \mathrm{SU}(2)$. In particular, they are connected.

Define complex-valued 1-forms on F by

$$\zeta_1 = \omega_2 + \mathrm{i}\,\omega_3\,, \quad \zeta_2 = \eta_2 - \mathrm{i}\,\eta_3\,, \quad \zeta_3 = \alpha_{21} + \mathrm{i}\,\alpha_{31}\,, \quad \zeta_4 = \beta_{21} - \mathrm{i}\,\beta_{31}\,.$$

These forms are λ-semibasic and satisfy the equations

$$\mathrm{d}\zeta_1 \equiv \cdots \mathrm{d}\zeta_4 \equiv 0 \mod \beta_{11}, \eta_1, \zeta_1, \ldots, \zeta_4\,,$$

while

$$\left.\begin{aligned} \mathrm{d}\beta_{11} &\equiv \zeta_3 \wedge \zeta_4 + \overline{\zeta_3} \wedge \overline{\zeta_4} \\ 2\,\mathrm{d}\eta_1 &\equiv \zeta_1 \wedge \zeta_4 - \zeta_2 \wedge \zeta_3 + \overline{\zeta_1} \wedge \overline{\zeta_4} - \overline{\zeta_2} \wedge \overline{\zeta_3} \end{aligned}\right\} \mod \beta_{11}, \eta_1\,. \tag{3.58}$$

Since the fibers of λ are connected, it follows that there is a (unique) complex structure $J : E \to E$ so that the complex-valued 1-forms on Λ that are $\mathbb{C}$-linear on E pull back to be linear combinations of $\beta_{11}, \eta_1, \zeta_1, \ldots, \zeta_4$. Moreover, the equations (3.58) imply that the almost CR-structure (E, J) is Levi-flat, as promised. This structure is clearly real-analytic since it is homogeneous under the action of the complex isometry group on Λ. (Note also, by the way, that the equations (3.58) also imply that this almost CR-structure is not integrable.) This completes the proof of Item 1.

Now suppose that $L \subset \mathbb{C}^3$ is a ruled special Lagrangian 3-fold that is not a 3-plane. Then, on a dense open set, this ruling can be chosen to be real-analytic and smooth. Consider the subbundle F_L of the adapted frame bundle over L that has $\mathbf{e}_1$ tangent to the ruling direction. Thus, the curves in L defined by the differential equations $\omega_2 = \omega_3 = 0$ are straight lines and, of course, $\mathbf{e}_1$ is tangent to these straight lines. It follows that $\mathrm{d}\mathbf{e}_1 \equiv 0 \mod \omega_2, \omega_3$. (In fact, this is necessary and sufficient that the $\mathbf{e}_1$-integral curves be straight lines in $\mathbb{C}^3$.) Since

$$\mathrm{d}\mathbf{e}_1 = \mathbf{e}_2\,\alpha_{21} + \mathbf{e}_3\,\alpha_{31} + J\mathbf{e}_1\,\beta_{11} + J\mathbf{e}_2\,\beta_{21} + J\mathbf{e}_3\,\beta_{31},$$

it follows, in particular, that $\beta_{11} \equiv \beta_{21} \equiv \beta_{31} \equiv 0 \mod \omega_2, \omega_3$. Since $\beta_{ij} = h_{ijk}\,\omega_k$, it follows from this that $h_{11j} = 0$ for $j = 1$, 2, and 3. In particular, the fundamental cubic

$$C = h_{ijk}\,\omega_i\omega_j\omega_k$$

is linear in the direction ω_1. Of course, by Proposition 3, it follows that, at points where C is non-zero, it is linear in at most three directions. Moreover, by Proposition 3 and Theorem 2, there is no non-planar special Lagrangian 3-fold whose cubic is linear in three directions. Thus, either C is linear in exactly two directions on a dense open set, or else it is linear in exactly one direction on a dense open set.

If C is linear in exactly two directions on a dense open set, then, again by Proposition 3, it follows that C is reducible at every point and, on a dense open set, cannot have an SO(3)-stabilizer isomorphic to S_3, since these are not linear in two distinct variables. It follows that the SO(3)-stabilizer at a generic point is $\mathbb{Z}_2$, so that, by Theorem 4, L must be one of the Lawlor-Harvey-Joyce examples. Moreover, the two linearizing directions, since they represent singular points of the projectivized cubic curve, must lie on the linear factor of C. Thus, the two possible rulings must lie in the 2-dimensional slices by 3-planes. Of course, this can only happen if the quadrics that are these slices are doubly ruled. Conversely, if the quadrics that are these slices are doubly ruled, then, obviously, L must be doubly ruled as well. This establishes Item 4 (as well as the fact that a non-planar special Lagrangian 3-fold cannot be triply ruled).

At any rate, note that $\beta_{11} = h_{11j}\,\omega_j = 0$ and that

$$\begin{aligned}\beta_{21} &= h_{212}\,\omega_2 + h_{213}\,\omega_3\,,\\ \beta_{31} &= h_{312}\,\omega_2 + h_{313}\,\omega_3 = h_{213}\,\omega_2 - h_{212}\,\omega_3\,,\end{aligned}$$

where I have used the symmetry and trace conditions on h_{ijk} together with the condition $h_{111} = 0$. It follows that

$$\big(\beta_{21} - \mathrm{i}\,\beta_{31}\big)\wedge\big(\omega_2 + \mathrm{i}\,\omega_3\big) = 0.$$

There are now two cases to deal with. Either β_{21} and β_{31} vanish identically or they do not.

Suppose first that $\beta_{21} \equiv \beta_{31} \equiv 0$. In this case, one can, after restricting to a dense open set, adapt frames so that the fundamental cubic has the form

$$C = h_{222}\,\big({\omega_2}^2 - 3\,\omega_2{\omega_3}^2\big),$$

where $h_{222} > 0$. In particular, the SO(3)-stabilizer of C at the generic point is S_3. Set $s = h_{222}$, so that the notation agrees with the notation established in §3.4. Looking back at the structure equations from that section, one sees that

$$\alpha_{21} + \mathrm{i}\,\alpha_{31} = -3(r_3 + \mathrm{i}\,r_2)(\omega_2 + \mathrm{i}\,\omega_3),$$

In particular, $(\alpha_{21} + \mathrm{i}\,\alpha_{31}){\wedge}(\omega_2 + \mathrm{i}\,\omega_3) = 0$. Since it has already been established that, in this case,

$$\beta_{11} = \eta_1 = \beta_{21} + \mathrm{i}\,\beta_{31} = \eta_2 + \mathrm{i}\,\eta_3 = 0,$$

it follows immediately that the natural map from the frame bundle to Λ that sends a coframe $u \in F_L$ to $\big(\mathbf{x}(u), \mathbf{e}_1(u)\big)$ maps the coframe bundle into an E-holomorphic surface and that this surface is simply the space of lines of the ruling.

Now suppose that β_{21} and β_{31} do not vanish identically. Then, by restricting to the dense open set where they are not simultaneously zero, we can reduce frames to arrange that $h_{212} = 0$, but that $h_{312} \neq 0$. In fact, there will exist functions $r \neq 0$, s, and t so that

$$C = 6r\,\omega_1\omega_2\omega_3 + s\big({\omega_2}^2 - 3\,\omega_2{\omega_3}^2\big) + t\big(3\,{\omega_2}^2\omega_3 - {\omega_3}^2\big).$$

This reduces the frames to a finite ambiguity, but I will not worry about this, since it does not impose any essential difficulty. Of course, s and t cannot vanish identically by Theorem 2. In particular, on this adapted bundle, the following formulae hold:

$$\begin{pmatrix}\beta_{11} & \beta_{12} & \beta_{13}\\ \beta_{21} & \beta_{22} & \beta_{23}\\ \beta_{31} & \beta_{32} & \beta_{33}\end{pmatrix} = \begin{pmatrix}0 & r\,\omega_3 & r\,\omega_2\\ r\,\omega_3 & s\,\omega_2 + t\,\omega_3 & r\,\omega_1 - s\,\omega_3 + t\,\omega_2\\ r\,\omega_2 & r\,\omega_1 - s\,\omega_3 + t\,\omega_2 & -s\,\omega_2 - t\,\omega_3\end{pmatrix}.$$

Now, there are functions p_{ij} and r_i, s_i, and t_i so that

$$\begin{aligned} dr &= r_1\,\omega_1 + r_2\,\omega_2 + r_3\,\omega_3\,,\\ ds &= s_1\,\omega_1 + s_2\,\omega_2 + s_3\,\omega_3\,,\\ dt &= t_1\,\omega_1 + t_2\,\omega_2 + t_3\,\omega_3\,,\end{aligned}$$

$$\begin{aligned} \alpha_{32} &= p_{11}\,\omega_1 + p_{12}\,\omega_2 + p_{13}\,\omega_3\,,\\ \alpha_{13} &= p_{21}\,\omega_1 + p_{22}\,\omega_2 + p_{23}\,\omega_3\,,\\ \alpha_{21} &= p_{31}\,\omega_1 + p_{32}\,\omega_2 + p_{33}\,\omega_3\,.\end{aligned}$$

Just as in previous cases of moving frame analyses, substituting these equations into the structure equations for $\mathrm{d}\beta_{ij}$ yields 15 equations on these 18 quantities. I will not give the whole solution, since that is not needed for this argument, but will merely note that these equations imply $p_{21} = p_{31} = 0$ and that $p_{22} = p_{33}$ while $p_{23} + p_{32} = 0$. In particular, this implies

$$(\alpha_{21} + \mathrm{i}\,\alpha_{31}) \wedge (\omega_2 + \mathrm{i}\,\omega_3) = 0,$$

just as in the first case. Moreover, since $\beta_{21} = r\,\omega_3$ and $\beta_{31} = r\,\omega_2$, it also follows that

$$(\beta_{21} - \mathrm{i}\,\beta_{31}) \wedge (\omega_2 + \mathrm{i}\,\omega_3) = 0.$$

Since it has already been shown that

$$\beta_{11} = \eta_1 = \eta_2 + \mathrm{i}\,\eta_3 = 0,$$

it follows, once again, that the natural map from the frame bundle to Λ that sends a coframe $u \in F_L$ to $\bigl(\mathbf{x}(u), \mathbf{e}_1(u)\bigr)$ maps the coframe bundle into an E-holomorphic surface and that this surface is simply the space of lines of the ruling.

Thus, it has been shown that any ruled special Lagrangian 3-fold is locally the 3-fold generated by an E-holomorphic surface in Λ.

The only thing left to check is that every E-holomorphic surface in Λ generates a special Lagrangian 3-fold in $\mathbb{C}^3$. However, given the analysis already done, this is an elementary exercise in the moving frame and can be safely left to the reader. $\square$

Remark 11 (The relation with special Lagrangian cones). It is not difficult to see that there is a Levi-flat almost CR-structure of codimension 1 on S^5 with the property that its holomorphic surfaces are exactly the links of special Lagrangian cones.

In fact, the mapping $\mathbf{e}_1 : \Lambda \to S^5$ is an almost CR-mapping in the obvious sense when S^5 is given this almost CR-structure. In particular, it follows that any ruled special Lagrangian 3-fold is associated to a special Lagrangian cone that one gets by simply translating all of the ruling lines so that they pass through one fixed point. It is in this sense that all of the ruled special Lagrangian 3-folds in $\mathbb{C}^3$ are 'twisted cones' in some sense.

In light of this fact, it may be that there is a formula for ruled special Lagrangian 3-folds that is analogous to the formula for austere 3-folds given in [6]. I have not yet tried to find this.

On the other hand, this relationship shows that there cannot be a 'Weierstrass formula' for the general ruled special Lagrangian like the formula given by Borisenko for the family that he discovered. The reason is that such a formula would, at the very least, imply a Weierstrass formula for the links of special Lagrangian cones. However, it is easy to show that this exterior differential system is equivalent to a Monge-Ampere system in 5-dimensions that, by a theorem of Lie, does not admit a Weierstrass formula. Thus, the best that one can hope for is Weierstrass formulae for special cases.

Remark 12 (The generalization to the associative case). As the reader may know, special Lagrangian 3-folds in $\mathbb{C}^3$ are special cases of a more general family of calibrated 3-folds in $\mathbb{R}^7$, namely, the *associative* 3-folds as described §IV of [12].

Regarding $\mathbb{R}^7$ as $\mathbb{R}\times\mathbb{C}^3$ and using x_0 as the standard linear coordinate on the $\mathbb{R}$-factor, the 3-form

$$\phi = dx_0 \wedge \left(\tfrac{\mathrm{i}}{2}\left(\mathrm{d}z_1{\wedge}\mathrm{d}\overline{z_1} + \mathrm{d}z_2{\wedge}\mathrm{d}\overline{z_2} + \mathrm{d}z_3{\wedge}\mathrm{d}\overline{z_3}\right)\right) + \mathrm{Re}(\mathrm{d}z_1\wedge \mathrm{d}z_2\wedge \mathrm{d}z_3) \tag{3.59}$$

is a calibration on $\mathbb{R}^7$, called the *associative calibration.* The 3-folds that it calibrates are said to be *associative.* The associative 3-folds that lie in the hyperplane $\{0\}\times\mathbb{C}^3$ are exactly the special Lagrangian 3-folds. However, there are many more associative 3-folds than special Lagrangian 3-folds, since, in particular, Harvey and Lawson prove that every connected real-analytic surface $S\subset\mathbb{R}^7$ lies in an (essentially) unique associative 3-fold [12, §IV.4, Theorem 4.1]

The subgroup of $\mathrm{GL}(7,\mathbb{R})$ that stabilizes ϕ is the compact exceptional group G_2. It acts transitively on the oriented lines in $\mathbb{R}^7$ through the origin, and the G_2-stabilizer of an oriented line is $\mathrm{SU}(3)$. In particular, the group Γ generated by the translations in $\mathbb{R}^7$ and the rotations in G_2 acts transitively on the space Λ of oriented lines in $\mathbb{R}^7$.

It can be shown that there is a unique Γ-invariant almost complex structure on Λ with the property that any pseudo-holomorphic surface $S\subset L$ defines a ruled associative 3-fold $\Sigma\subset\mathbb{R}^7$ (i.e., the oriented union of the oriented lines in $\mathbb{R}^7$ that the points of S represent) and, conversely, that if $\Sigma\subset\mathbb{R}^7$ is a ruled associative 3-fold that is not a 3-plane, then the set of oriented lines $S\subset\Lambda$ that meet Σ in at least an interval is a pseudo-holomorphic curve in Λ.

Further development of this description allows one to give a description of the ruled associative 3-folds of $\mathbb{R}^7$ that directly generalizes Joyce's description in [16] of the ruled special Lagrangian 3-folds in $\mathbb{C}^3$.

The details of these results will be reported on elsewhere.

References

[1] A. Borisenko, *Ruled special Lagrangian surfaces*, Minimal surfaces, Amer. Math. Soc., Providence, RI, 1993, pp. 269–285. 4, 29

[2] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths, *Exterior differential systems*, Springer-Verlag, New York, 1991. 16, 29

[3] Robert L. Bryant, *Bochner-Kahler metrics*, arXiv:math.DG/0003099. 24

[4] ———, *Calibrated embeddings in the special Lagrangian and coassociative cases*, DUKE-CGTP-99-09, arXiv:math.DG/9912246. 2

[5] Robert L. Bryant, *Minimal Lagrangian submanifolds of Kähler-Einstein manifolds*, Differential geometry and differential equations (Shanghai, 1985), Springer, Berlin, 1987, pp. 1–12. 6

[6] ———, *Some remarks on the geometry of austere manifolds*, Bol. Soc. Brasil. Mat. (N.S.) **21** (1991), no. 2, 133–157. 17, 17, 17, 18, 18, 35

[7] ———, *Some examples of special Lagrangian tori*, Adv. Theor. Math. Phys. **3** (1999), no. 1, 83–90, arXiv:math.DG/9902076. 2

[8] Edward Goldstein, *Calibrated Fibrations on Complete Manifolds via Torus Action*, arXiv:math.DG/0002097. 5

[9] ———, *Special Lagrangian submanifolds and Algebraic complexity one Torus Actions*, arXiv:math.DG/0003220. 2

[10] Mark Gross, *Examples of Special Lagrangian Fibrations*, arXiv:math.AG/0012002. 5

[11] F. Reese Harvey, *Spinors and calibrations*, Academic Press Inc., Boston, MA, 1990. 2, 4, 22, 25

[12] Reese Harvey and H. Blaine Lawson, Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157. 2, 4, 5, 5, 5, 13, 21, 29, 29, 29, 35, 35

[13] Mark Haskins, *Special Lagrangian Cones*, arXiv:math.DG/0005164. 2, 4, 16, 26

[14] Nigel J. Hitchin, *The moduli space of special Lagrangian submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), no. 3-4, 503–515 (1998), arXiv:dg-ga/9711002, Dedicated to Ennio De Giorgi. 2

[15] Dominic Joyce, *On counting special Lagrangian homology 3-spheres*, arXiv:hep-th/9907013. 2

[16] ———, *Ruled special Lagrangian 3-folds in* $\mathbb{C}^3$, arXiv:math.DG/0012060. 29, 35

[17] ———, *Special Lagrangian m-folds in* $\mathbb{C}^m$ *with symmetries*, arXiv:math.DG/0008021. 5

[18] Gary Lawlor, *The angle criterion*, Invent. Math. **95** (1989), no. 2, 437–446. 22

[19] ———, *Pairs of planes which are not size-minimizing*, Indiana Univ. Math. J. **43** (1994), no. 2, 651–661. 2

[20] Naichung Conan Leung, Shing-Tung Yau, and Eric Zaslow, *From Special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai Transform*, arXiv:math.DG/0005118. 2

[21] Robert C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** (1998), no. 4, 705–747. 2, 5

[22] Sema Salur, *Deformations of Special Lagrangian Submanifolds*, arXiv:math.DG/9906048. 5

[23] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), no. 1-2, 243–259, arXiv:hep-th/9606040. 2

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